The Free Algebra in a Two-sorted Variety of State Algebras

Tomáš Kroupa[1]* and Vincenzo Marra[2]

[1] Institute of Information Theory and Automation, Czech Academy of Sciences
Prague, Czech Republic
kroupa@utia.cas.cz

[2] Dipartimento di Matematica, Università degli Studi di Milano
Milano, Italy
Vincenzo.Marra@unimi.it

States of MV-algebras [6] are $[0,1]$-valued functions, which generalise finitely-additive probability measures on boolean algebras, and whose domains are MV-algebras [2]. Flaminio and Montagna [3] introduced an internal state as an additional unary operation $\sigma\colon M \to M$ satisfying certain equational laws on an MV-algebra M. Internal states capture the basic properties of states in a setting amenable to universal-algebraic techniques.

In our note [5] we made first steps towards a general two-sorted algebraic model for expressing the notion of state between two MV-algebras M and N, making thus a fundamental distinction between events (captured by elements of the domain M) and probability degrees (represented by the co-domain N). A *generalised state* of M with values in N is a mapping $s\colon M \to N$ such that for every $a, b \in M$ the following hold: $s(a \oplus b) = s(a) \oplus s(b \wedge \neg a)$, $s(\neg a) = \neg s(a)$, and $s(\top) = \top$. A *state algebra* is a two-sorted algebra (M, N, s), where the operations of M and N are in the single sorts given by M and N, respectively, and the only operation between the two sorts is the generalised state s. The class of all state algebras constitutes a two-sorted algebraic variety. Most universal-algebraic constructions and results have analogous correspondents in the multi-sorted setting [1].

In this contribution we will characterise the free state algebra $\mathbf{F}(S_1, S_2)$ generated by a two-sorted set of generators (S_1, S_2). The free state algebra can be expressed as

$$\mathbf{F}(S_1, S_2) = \mathbf{F}(S_1, \emptyset) \amalg \mathbf{F}(\emptyset, S_2),$$

where $\mathbf{F}(S_1, \emptyset)$ and $\mathbf{F}(\emptyset, S_2)$ are the free state algebras over $(S_1, \emptyset)$ and $(\emptyset, S_2)$, respectively, and $\amalg$ denotes the coproduct operation in the multi-sorted algebraic category of state algebras. First, the algebra $\mathbf{F}(S_1, \emptyset)$ is isomorphic to the state algebra $(F(S_1), \langle \widehat{F(S_1)} \rangle, \alpha)$, where $F(S_1)$ is the free MV-algebra over S_1, $\langle \widehat{F(S_1)} \rangle$ is the affine representation of $F(S_1)$ (see [4]), and α is the evaluation map $F(S_1) \to \langle \widehat{F(S_1)} \rangle$ sending elements of $F(S_1)$ to $[0,1]$-valued affine functions over the state space of $F(S_1)$. Second, the free state algebra over $(\emptyset, S_2)$ is $\mathbf{F}(\emptyset, S_2) = (\mathbb{2}, F(S_2), s_0)$, where $\mathbb{2}$ is the two-element MV-algebra, $F(S_2)$ is the free MV-algebra generated by S_2, and s_0 is the only possible generalised state $\mathbb{2} \to F(S_2)$. We will show that

$$\mathbf{F}(S_1, \emptyset) \amalg \mathbf{F}(\emptyset, S_2) = (F(S_1), \langle \widehat{F(S_1)} \rangle \amalg_{MV} F(S_2), \beta_1 \circ \alpha),$$

where

$$\langle \widehat{F(S_1)} \rangle \amalg_{MV} F(S_2)$$

is the coproduct (free product [7]) of MV-algebras $\langle \widehat{F(S_1)} \rangle$ and $F(S_2)$, and the map $\beta_1\colon \langle \widehat{F(S_1)} \rangle \to \langle \widehat{F(S_1)} \rangle \amalg_{MV} F(S_2)$ is the coproduct injection.

*The first author was supported by the GAČR project n. 17-04630S.

References

[1] G. Birkhoff and J. D. Lipson. Heterogeneous algebras. *Journal of Combinatorial Theory*, 8:115–133, 1970.

[2] R. L. O. Cignoli, I. M. L. D'Ottaviano, and D. Mundici. *Algebraic foundations of many-valued reasoning*, volume 7 of *Trends in Logic—Studia Logica Library*. Kluwer, Dordrecht, 2000.

[3] T. Flaminio and F. Montagna. MV-algebras with internal states and probabilistic fuzzy logics. *International Journal of Approximate Reasoning*, 50(1):138–152, 2009.

[4] K. R. Goodearl. *Partially ordered abelian groups with interpolation*, volume 20 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1986.

[5] T. Kroupa and V. Marra. Generalised states: a multi-sorted algebraic approach to probability. *Soft Computing*, 21(1):57–67, 2017.

[6] D. Mundici. Averaging the truth-value in Łukasiewicz logic. *Studia Logica*, 55(1):113–127, 1995.

[7] D. Mundici. *Advanced Łukasiewicz calculus and MV-algebras*, volume 35 of *Trends in Logic*. Springer, 2011.