Disjunction and Existence Property in Inquisitive Logic

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The system `InqBQ` ([1], [4], [3]) generalizes `FOL` (first order classical logic) to study *dependencies* between `FOL` structures in a similar fashion to Dependence Logic ([5]) and other logics based on team semantics ([2], [6]). In this paper we introduce several model theoretic constructions useful to study the entailment relation of `InqBQ`, and prove the disjunction and existence for the classical fragment of the logic (presented in [3]).

GENERALIZING `FOL` SEMANTICS

In the rest of the paper with $\Sigma = \{f, \ldots ; R, \ldots\}$ we indicate a fixed `FOL` signature.

Definition (Skeleton of a model)**.** Given $M = \langle D^M; f^M, \ldots ; R^M, \ldots ; \sim^M \rangle$ a `FOL` structure (note that we introduce here an extensional equality $\sim^M$, i.e. a congruence wrt $f^{\mathcal{M}}$ and $R^{\mathcal{M}}$) we define its **skeleton** as the tuple $\mathrm{Sk}(M) = \langle D^M; f^M, \ldots \rangle$ consisting of the domain and the interpretation of the function symbols.

Definition (Information model)**.** An **information model** is a tuple $\mathcal{M} = \langle M_w | w \in W^{\mathcal{M}} \rangle$ where the M_w are `FOL` structures sharing the same skeleton: $\forall w, w'. \mathrm{Sk}(M_w) = \mathrm{Sk}(M_{w'})$.
Conceptually, an information model represents a *collection of possible states of affairs* and we can represent a body of information by selecting the structures compatible with it.

Definition (Info state)**.** Given a model $\mathcal{M}$ we call a subset of the structures that compose it an **info state**: $s \subseteq W$ (modulo a natural identification). We call a model $\mathcal{M}_s = \langle M_w | w \in s \rangle$ for $s \subseteq W^{\mathcal{M}}$ a **submodel of** $\mathcal{M}$.

Definition (Support semantics)**.** Let $\mathcal{M}$ be a model, s an info state of $\mathcal{M}$ and $g : \mathrm{Var} \to D^{\mathcal{M}}$ a valuation. Let α be a `FOL` formula. We define the support relation by the following inductive clauses. As a notational convention, we will omit s if $s = W^{\mathcal{M}}$.

$$\begin{aligned}
\mathcal{M}, s \vDash_g \bot &\iff s = \emptyset \\
\mathcal{M}, s \vDash_g [t_1 = t_2] &\iff \forall w \in s. M_w \vDash_g^{\mathtt{FOL}} [t_1 = t_2] \\
\mathcal{M}, s \vDash_g R(\bar{t}) &\iff \forall w \in s. M_w \vDash_g^{\mathtt{FOL}} R(\bar{t}) \\
\mathcal{M}, s \vDash_g \phi \wedge \psi &\iff \mathcal{M}, s \vDash_g \phi \text{ and } \mathcal{M}, s \vDash_g \psi \\
\mathcal{M}, s \vDash_g \phi \to \psi &\iff \forall t \subseteq s. \text{ if } \mathcal{M}, t \vDash_g \phi \\
&\qquad \text{then } \mathcal{M}, t \vDash_g \psi \\
\mathcal{M}, s \vDash_g \forall x. \phi &\iff \forall d \in D^{\mathcal{M}}. \mathcal{M}, s \vDash_{g[x \mapsto d]} \phi
\end{aligned}$$

We say that a theory Γ entails a formula α (notation $\Gamma \vDash \alpha$) if and only if for every tuple $\langle \mathcal{M}, s, g \rangle$ that supports Γ, this supports also α.

Lemma (Properties of the support semantics for `FOL` formulas)**.**
Flatness: $\mathcal{M}, s \vDash_g \alpha \iff \forall w \in s.\ \mathcal{M}, \{w\} \vDash_g \alpha$.
Classical World Support: $\mathcal{M}, \{w\} \vDash_g \alpha \iff M_w \vDash_g^{\mathtt{FOL}} \alpha$.
Classical Validity Preservation: $\Gamma \vDash \alpha \iff \Gamma \vDash^{\mathtt{FOL}} \alpha$.

ADDING NEW OPERATORS TO THE LOGIC

Defined this generalized semantics, we can now introduce new logical operators to describe *connections and relations between models sharing the same skeleton*. We consider here **the logic** `InqBQ` obtained by adding the **operator** $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and the **quantifier** $\overline{\exists}$, and their associated semantical clauses.

$$\mathcal{M}, s \vDash_g \phi \vee\!\!\!\vee \psi \iff \mathcal{M}, s \vDash_g \phi \text{ or } \mathcal{M}, s \vDash_g \psi$$
(a disjunct holds at the *whole state*)
$$\mathcal{M}, s \vDash_g \overline{\exists} x. \phi \iff \exists d \in D^{\mathcal{M}}.\ \mathcal{M}, s \vDash_{g[x \mapsto d]} \phi$$
(an element is a *uniform* witness of ϕ at s)

Lemma (**Downward Closure**). $\mathcal{M}, s \vDash_g \phi$ and $t \subseteq s$, then $\mathcal{M}, t \vDash_g \phi$.
Note that *flatness holds exactly for those formulas ϕ which are semantically equivalent to a* `FOL` *formula* ([3]).

DISJUNCTION AND EXISTENCE PROPERTIES

Theorem. Let Γ be a `FOL` theory and α a `FOL` formula. Then:
Disjunction Property: If $\Gamma \vDash \phi \mathbin{\vee\!\!\!\vee} \psi$, then $\Gamma \vDash \phi$ or $\Gamma \vDash \psi$.
Existence Property: If $\Gamma \vDash \overline{\exists} x.\phi(x)$, then $\Gamma \vDash \phi(t)$ for some term t.

The proof of this theorem is based on the introduction of some relevant model-theoretic constructions.
The $\oplus$ operator: We can define an operator $\oplus$ such that, given a set of models $\{\mathcal{M}_i | i \in I\}$, it produces a model $\oplus_{i \in I} \mathcal{M}_i$ with the following properties

$$\mathcal{M}_i \nvDash \phi \implies \oplus_{i \in I} \mathcal{M}_i \nvDash \phi \qquad\qquad \forall i \in I.\ \mathcal{M}_i \vDash \alpha \iff \oplus_{i \in I} \mathcal{M}_i \vDash \alpha \text{ for } \alpha \text{ classical}$$

This construction strongly relies on *downward closure* and *flatness* for classical formulas.
Characteristic model of Γ: Consider a classical theory Γ. For every non entailment $\Gamma \nvDash \phi$ we can select a model $\mathcal{M}_\phi$ that is a *witness* of it, meaning $\mathcal{M}_\phi \vDash_{g_\phi} \Gamma$ and $\mathcal{M}_\phi \nvDash_{g_\phi} \phi$. If we define now $\mathcal{M}_\Gamma = \oplus_{\Gamma \nvDash \phi} \mathcal{M}_\phi$, by the property of $\oplus$ we obtain $\mathcal{M}_\Gamma \vDash \psi \iff \Gamma \vDash \psi$.
Note that this model can be used to easily prove the disjunction property:

$$\Gamma \vDash \phi \mathbin{\vee\!\!\!\vee} \psi \iff \mathcal{M}_\Gamma \vDash \phi \mathbin{\vee\!\!\!\vee} \psi \iff \mathcal{M}_\Gamma \vDash \phi \text{ or } \mathcal{M}_\Gamma \vDash \psi \iff \Gamma \vDash \phi \text{ or } \Gamma \vDash \psi$$

Blow-up model: given a model $\mathcal{M}$ we can define an elementarily equivalent model $\mathcal{BM}$ (the **blow-up of** $\mathcal{M}$) whose domain is $\mathcal{T}\Sigma(D^{\mathcal{M}})$, the free algebra of terms in the extended signature $\Sigma(D^{\mathcal{M}})$ obtained by adding to Σ a fresh constant symbol for every element of $D^{\mathcal{M}}$. In this step, the intensional equality plays a fundamental role.
Permutation models: given a model $\mathcal{M}$ and a permutation of its elements $\sigma \in \mathfrak{S}\left(D^{\mathcal{M}}\right)$, we can naturally extend such permutation to $\mathcal{T}\Sigma\left(D^{\mathcal{M}}\right)$. Using this, we can define a model $\mathcal{B}^\sigma\mathcal{M}$ by *permuting the names of the elements of $\mathcal{BM}$ according to σ*.
Note that this operation *preserves skeletons* ($\mathrm{Sk}(\mathcal{BM}) = \mathrm{Sk}(\mathcal{B}^\sigma\mathcal{M})$) and that *a closed term t of $\Sigma\left(D^{\mathcal{M}}\right)$ is fixed under every permutation σ iff t is a closed term of* Σ. These two properties (of great importance for the proof of the existence property) wouldn't hold if the permutation was applied directly to $\mathcal{M}$, thus the necessity of defining the model $\mathcal{BM}$.
Lemma. $\mathcal{B}^\sigma\mathcal{M} \vDash \phi(\sigma(d_1), \dots, \sigma(d_n)) \iff \mathcal{BM} \vDash \phi(d_1, \dots, d_n) \iff \mathcal{M} \vDash \phi(d_1, \dots, d_n)$
The full permutation model: Consider now the model $\mathcal{M}_\Gamma$. As the action of a permutation $\sigma \in \mathfrak{S}\left(D^{\mathcal{M}_\Gamma}\right)$ preserves the skeleton of $\mathcal{B}(\mathcal{M}_\Gamma)$, we can consider the new model $\mathfrak{S}\left(\mathcal{M}_\Gamma\right) = \{M | \exists\sigma.\ M \in \mathcal{B}^\sigma\left(\mathcal{M}_\Gamma\right)\}$. By building $\mathcal{M}_\Gamma$ in a suitable way, we can obtain the following two properties:
$\underline{\mathfrak{S}\,\mathcal{M}_\Gamma \vDash \phi \iff \Gamma \vDash \phi}$: since $\mathfrak{S}\left(\mathcal{M}_\Gamma\right) \vDash \Gamma$ can be tested on single worlds by *flatness*, and $\mathcal{B}\left(\mathcal{M}_\Gamma\right)$ is a submodel of $\mathfrak{S}\left(\mathcal{M}_\Gamma\right)$.
$\underline{\mathfrak{S}\,\mathcal{M}_\Gamma \vDash \overline{\exists} x.\phi(x) \Rightarrow \mathfrak{S}\,\mathcal{M}_\Gamma \vDash \phi(t) \text{ for some closed } t}$: the intuitive reason being that the role of two elements can be *swapped* in the model $\mathfrak{S}\left(\mathcal{M}_\Gamma\right)$ as long as they are not fixed by every permutation σ. From this we obtain that, if there exists an element without the property ϕ, then every element that is not the interpretation of a closed term of Σ does not have the property.
Note that this model can be used to easily prove the existence property:

$$\Gamma \vDash \overline{\exists} x.\phi(x) \iff \mathfrak{S}\left(\mathcal{M}_\Gamma\right) \vDash \overline{\exists} x.\phi(x) \iff \exists t \text{ closed}.\ \mathfrak{S}\left(\mathcal{M}_\Gamma\right) \vDash \phi(t) \iff \exists t \text{ closed}.\, \Gamma \vDash \phi(t)$$

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