Admissibility for universal classes and multi-conclusion consequence relations

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The notions of admissibility and structural completeness for logics and consequence relations has received considerable attention for many years. Recently a study of these concepts has been undertaken by Rosalie Iemhoff [1]. It appears that admissibility may be considered in various nonequivalent ways. This leads to variants of structural completeness. Here we investigate this topic from an algebraic perspective. We provide algebraic characterizations of variants of structural completeness for universal classes (which are algebraic counterparts of multi-conclusion consequence relations). Then we study the preservation of these properties by the Blok-Esakia isomorphism.

A (multi-conclusion) *rule* is an ordered pair, written as Γ/Δ, of finite sets of formulas in a given propositional language. When $|\Delta| = 1$ we talk about a *single-conclusion rule*. A set of rules, written as a relation $\vdash$, is a *multi-conclusion consequence relation* (*mcr*) if for all finite sets $\Gamma, \Gamma', \Delta, \Delta'$ of formulas, for every formula φ and for every substitution s the following holds

- $\varphi \vdash \varphi$;
- if $\Gamma \vdash \Delta, \varphi$ and $\Gamma, \varphi \vdash \Delta$, then $\Gamma \vdash \Delta$;
- if $\Gamma \vdash \Delta$, then $\Gamma, \Gamma' \vdash \Delta, \Delta'$;
- if $\Gamma \vdash \Delta$, then $s(\Gamma) \vdash s(\Delta)$.

(We omit the curly brackets for sets, write commas for unions and omit the empty set.)

Informally speaking, admissible rules are rules that may be added to a mcr in order to improve the search of (multi)theorems. A formal definition of admissible rules for the basic and narrow variants was given by Rosalie Iemhoff in [1] (called there *full* and *strict*). The weak variant is taken from [3]. For simplicity, we consider the admissibility for single-conclusion rules.

Definition 1. For a rule $r = \Gamma/\delta$ and a mcr $\vdash$ let $\vdash_r$ be a least mcr extending $\vdash$ and containing r. Then r is

- *admissible for* $\vdash$ provided $\vdash \Delta$ iff $\vdash_r \Delta$ for every finite set Δ of formulas;
- *weakly admissible for* $\vdash$ provided $\vdash \varphi$ iff $\vdash_r \varphi$ for formula φ;
- *narrowly admissible for* $\vdash$ provided for every substitution s $(\forall \gamma \in \Gamma\ \vdash s(\gamma))$ yields $\vdash s(\delta)$.

And $\vdash$ is (*strongly*, *widely*) *structurally complete* if every (weakly, narrowly) admissible for $\vdash$ single-conclusion rule belongs to $\vdash$.

Fact 2. *For a single-conclusion rule r and a mcr $\vdash$ we have the implications:*

r is admissible for $\vdash$ $\Rightarrow$ r is weakly admissible for $\vdash$ $\Rightarrow$ r is narrowly admissible for $\vdash$,
$\vdash$ is widely struct. complete $\Rightarrow$ $\vdash$ is strongly struct. complete $\Rightarrow$ $\vdash$ is struct. complete.

Algebraic counterparts of single conclusion consequence relations are quasivarieties of algebras. A main tool to deal with the admissibility is then the notion of free algebras for quasivarieties: the admissibility corresponds exactly to the validity on free algebras.

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Algebraic counterparts of mcrs are universal classes of algebras[1]. Clearly, for every universal clas $\mathcal{U}$ free algebras exist. But they do not have to belong to $\mathcal{U}$. We overcome this obstacle by introducing the notion of a free family. It consists of quotients of a term algebra chosen in a certain minimal way [4]. Thus this object is indeed similar to a free algebra.

Recall that a counterpart of a single-conclusion rule is a quasi-identity. We skip definitions of the variants of the admissibility for quasi-identities. Let us just note that they are direct translations of Definition 1 [4]. In order to formulate our theorem let us introduce a bit of notation: For a universal class $\mathcal{U}$ let $\mathbf{F}$ be its free algebra and $\mathcal{F}$ be its free family, both of denumerable rank. For a class $\mathcal{K}$ of algebras let $\mathsf{Q}(\mathcal{K})$ be a least quasivariety containing $\mathcal{K}$.

Theorem 3. *Let q be a quasi-identity and $\mathcal{U}$ be a universal class. Then*

- *q is admissible for $\mathcal{U}$ iff $\mathcal{F} \models q$;*
- *q is weakly admissible for $\mathcal{U}$ iff $\mathbf{F} \in \mathsf{Q}(\{\mathbf{A} \in \mathcal{U} \mid \mathbf{A} \models q\})$;*
- *q is narrowly admissible for $\mathcal{U}$ iff $\mathbf{F} \models q$.*

Consequently,

- *$\mathcal{U}$ is structurally complete iff $\mathsf{Q}(\mathcal{F}) = \mathsf{Q}(\mathcal{U})$;*
- *$\mathcal{U}$ is strongly structurally complete iff $\mathbf{F} \in \mathsf{Q}(\mathcal{U} \cap \mathcal{Q})$ yields $\mathcal{U} \subseteq \mathcal{Q}$ for every quasivariety $\mathcal{Q}$;*
- *$\mathcal{U}$ is widely structurally complete iff $\mathsf{Q}(\mathbf{F}) = \mathsf{Q}(\mathcal{U})$.*

The classical Blok-Esakia theorem states that there is one to one correspondence between extensions of intuitionistic logic and normal extensions of modal Grzegorczyk logic. In [2] Emil Jeřábek extended this fact to mcrs. Algebraically it says that there is an isomorphism σ from the lattice of universal classes of Heyting algebras onto the lattice of universal classes of modal Grzegorczyk algebras (see [3] for an algebraic treatment of this topic). In [3, 4] we obtained the following preservation and reflection facts.

Theorem 4. *Let $\mathcal{U}$ be a universal class of Heyting algeras. Then*

- *$\mathcal{U}$ is structurally complete iff $\sigma(\mathcal{U})$ is structurally complete;*
- *$\mathcal{U}$ is strongly structurally complete iff $\sigma(\mathcal{U})$ is strongly structurally complete;*
- *$\mathcal{U}$ is widely structurally complete iff $\sigma(\mathcal{U})$ is widely structurally complete.*

Let us finish with the remark that the admissibility and the weak admissibility properties may be directly defined also for multi-conclusion rules. Then the analogs of the presented results may be proved. However there is a problem with the narrow variant of the admissibility. It may be defined in at least two different ways. The connection to the strong and the basic variants is more complex.

References

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[1] However, at this moment there is no general theory on algebraization of mcrs