Unifying the Leibniz and Maltsev hierarchies

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Universal algebra and abstract algebraic logics are two theories that study, respectively, arbitrary algebraic structures and arbitrary substitution-invariant consequence relations (sometimes called deductive systems). The interplay between the two theories can be hardly overestimated. On the one hand, techniques from universal algebra have been fruitfully applied to the study of propositional logics in the framework of abstract algebraic logic. On the other hand, any class of algebras K is naturally associated with a substitution-invariant equational consequence $\vDash_{\mathsf{K}}$ (representing the validity of generalized quasi-equations in K), which is amenable to the techniques of abstract algebraic logic. The fact that universal algebra and abstract algebraic logic pursue two tightly connected paths is nicely reflected in the fact that one of the main achievements of both theories is a taxonomy in which, respectively, varieties and deductive systems are classified. In universal algebra, this taxonomy is called *Maltsev hierarchy*, while in abstract algebraic logic it is known as *Leibniz hierarchy*.

The goal of this contribution is to show that this analogy between the Maltsev and Leibniz hierarchies can be made mathematically precise, in a such way that the traditional Maltsev hierarchy coincides with the restriction of a suitable *finite companion* of the Leibniz hierarchy formulated for two-deductive systems. To this end, we need to solve a fundamental asymmetry between the theories of the Maltsev and Leibniz hierarchy: while there is a precise definition of what the Maltsev hierarchy is [3, 4, 5], no such agreement exists for the case of the Leibniz hierarchy.

For the sake of simplicity, we will introduce the main new definitions for *logics*, i.e. substitution-invariant consequence relations formulated over the set of formulas (built up with an arbitrarily large infinite set of variables) of an algebraic language. Recall that each logic $\vdash$ is naturally associated with a class of matrices $\mathsf{Mod}^{\mathrm{Su}}(\vdash)$, called the *Suszko models* of $\vdash$ [1]. An *interpretation* of a logic $\vdash$ into a logic $\vdash'$ is a map $\boldsymbol{\tau}$ assigning an n-ary term $\boldsymbol{\tau}(f)$ of $\vdash'$ to every n-ary connective f of $\vdash$ in such a way that

$$\text{if } \langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash'), \text{ then } \langle \boldsymbol{A}^{\boldsymbol{\tau}}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash)$$

where $\boldsymbol{A}^{\boldsymbol{\tau}}$ is the algebra in the language of $\vdash$, whose universe is A and in which the connective f is interpreted as the term-function $\boldsymbol{\tau}(f)^{\boldsymbol{A}}$ of $\boldsymbol{A}$. We write $\vdash \leq \vdash'$ to denote the fact that $\vdash$ is interpretable into $\vdash'$. The interpretability relation $\leq$ is a preorder on the class of all logics. We denote by Log the poset obtained identifying equi-interpretable logics.

Theorem 1. Log *is a complete meet-semilattice, meaning that infima of all its subsets exist. Moreover,* Log *is not a join-semilattice. Finally,* Log *has no minimum element, it has a maximum and a coatom (that under Vopěnka's Principle is unique).*

A *Leibniz condition* is a sequence $\Phi \coloneqq \{\vdash_\alpha : \alpha \in \mathsf{Ord}\}$ of logics indexed by all ordinals Ord, satisfying the following additional condition: if $\alpha \leq \beta$, then $\vdash_\beta \leq \vdash_\alpha$. The class of *models* of Φ is $\mathsf{Mod}(\Phi) \coloneqq \{\vdash : \vdash_\alpha \leq \vdash \text{ for some } \alpha \in \mathsf{Ord}\}$. A *Leibniz class* is a class of logics M for which there is a Leibniz condition Φ such that $\mathsf{M} = \mathsf{Mod}(\Phi)$. It is not difficult to see that all classes of logics traditionally included into the Leibniz hierarchy are in fact Leibniz classes in this general sense. For this reason, we propose to identify the Leibniz hierarchy with the poset of all Leibniz classes. Leibniz classes can be characterized in terms of closure under certain constructions, that we call *Taylorian products* and *compatible expansions*, as follows (cf. [4, 5]):

Theorem 2. *Let* M *be a class of logics. The following conditions are equivalent:*

1. M *is a Leibniz class.*
2. M *is closed under term-equivalence, compatible expansions and Taylorian products.*
3. M *is a complete filter of* Log.

The fact that Leibniz classes can be identified with complete filters of Log rises the question of understanding which of the classical Leibniz classes determine a meet-irreducible or prime filter (cf. [2]). This is a completely new direction of research. Nevertheless, we were able to obtain some promising results: for example, it turns out that, in the setting of logics with theorems, the class of equivalential logics is meet-reducible, while (under the assumption of Vopěnka's Principle) the classes of truth-equational and assertional logics are prime.

As we mentioned, it is possible to associate a *finite companion* to the Leibniz hierarchy, understood as the poset of all Leibniz classes. Roughly speaking, this is the collection of Leibniz classes determined by Leibniz conditions of the form $\Phi = \{\vdash_n : \alpha \in \omega\}$, where $\vdash_n$ is a finitely presentable and finitely equivalential logic. We call *finitely presentable Leibniz classes* the classes of logics in the finite companion of the Leibniz hierarchy. The Maltsev hierarchy is then the restriction of the finite companion of the Leibniz hierarchy of two-deductive system to equational consequences. More precisely, we have the following:

Theorem 3. *Let* K *be a class of varieties.* K *is a Maltsev class iff there is a finitely presentable Leibniz class* M *of two-deductive systems such that* $\mathsf{K} = \{\mathsf{V} : \mathsf{V}$ *is a variety and* $\vDash_{\mathsf{V}} \in \mathsf{M}\}$.

The above result shows that the logical theory of the Leibniz hierarchy may be seen as a generalization of the algebraic theory of Maltsev classes. Moreover, in our opinion, this perspective shows that the conceptual taxonomies, which lie at the heart of modern abstract algebraic logic and universal algebra, have a common root.

References

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