The computational complexity of the Leibniz hierarchy

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Abstract algebraic logic is a field that studies uniformly propositional logics [2, 3, 4]. One of its main achievements is the development of the so-called *Leibniz hierarchy* (see Figure 1), which provides a taxonomy that classifies propositional systems accordingly to the way their notions of *logical equivalence* and of *truth* can be defined.

A fundamental question, that arose in the study of the Leibniz hierarchy, is whether there is an algorithm that allows to classify logics in the Leibniz hierarchy. The answer to this question depends on the way in which these logics are presented. More precisely, in [7] it is shown that the problem of classifying logics presented *syntactically*, i.e. by means of finite Hilbert calculi, in the Leibniz hierarchy is in general undecidable. On the other hand, it is not difficult to see that logics presented *semantically*, i.e. by means of finite sets of finite (logical) matrices of finite type, can be classified mechanically in the Leibniz hierarchy. It is therefore natural to ask which is the computational complexity of the problem of classifying semantically presented logics in the Leibniz hierarchy. More precisely, in this contribution we will present a solution to the following problems:

- Let K be a level of the Leibniz hierarchy. Which is the computational complexity of the problem $\mathsf{Class\text{-}K}$ of determining whether a semantically presented logic belongs to K?

Elementary considerations show that the naive algorithms, that solve $\mathsf{Class\text{-}K}$, run in exponential time. The interesting part of our proof consists in establishing a hardness result, according to which these algorithms cannot be substantially improved. In [1] it was established that the following problem, which we denote by $\mathsf{Gen\text{-}Clo}^1_2$, is complete for **EXPTIME**:

- Let $\boldsymbol{A}$ be a finite algebra of finite type, whose basic operations are at most binary, and a h be a unary function on A. Does h belong to the clone of $\boldsymbol{A}$?

We will construct a polynomial-time reduction of $\mathsf{Gen\text{-}Clo}^1_2$ to $\mathsf{Class\text{-}K}$.

To this end, consider a non-trivial algebra $\boldsymbol{A}$ whose basic operations $\mathcal{F}$ are at most binary, and a unary function h on A. For sake of simplicity, we assume that $\mathcal{F}$ contains no constant symbols. Our goal is to define a new algebra $\boldsymbol{A}^\natural$, related to $\boldsymbol{A}$ and h. The construction of the $\boldsymbol{A}^\natural$ is partially reniniscent of ideas exploited in [6] and [5] to prove some hardness results related to type sets and Maltsev conditions. The universe of $\boldsymbol{A}^\natural$ is given by eight disjoint copies $A_1, \dots, A_8$ of A. Given an element $a \in A$, we will denote by a^i its copy in A_i. The basic operation of $\boldsymbol{A}^\natural$ are the ones in $\mathcal{F}$ plus a new ternary operation $\heartsuit$ and a new unary operation $\Box$. Their interpretation is defined as follows. Given an n-ary operation $f \in \mathcal{F}$ and $a_1^{m_1} \dots, a_n^{m_n} \in A^\natural$, we set

$$f(a_1^{m_1} \dots, a_n^{m_n}) \coloneqq f^{\boldsymbol{A}}(a_1, \dots, a_n)^5.$$

Observe that all the operations $f^{\boldsymbol{A}^\natural}$ with $f \in \mathcal{F}$ give values in A_7. Given $a^m, b^n, c^k \in A^\natural$, we set

$$\heartsuit(a^m, b^n, c^k) \coloneqq \begin{cases} a^1 & \text{if } a^m = c^k \text{ and } h(a)^5 = b^n \text{ and } m \in \{1,3,4\} \\ a^2 & \text{if } a^m = c^k \text{ and } h(a)^5 = b^n \text{ and } m \in \{2,5,6,7,8\} \\ a^4 & \text{if } m, k \in \{1,3,4\} \text{ and (either } a^m \neq c^k \text{ or } h(a)^5 \neq b^n) \\ a^7 & \text{if } \{m,k\} \cap \{2,5,6,7,8\} \neq \emptyset \text{ and} \\ & \text{(either } a^m \neq c^k \text{ or } h(a)^5 \neq b^n). \end{cases}$$

$$\begin{tikzcd}[column sep=small] & \text{algebraizable} \arrow[dl] \arrow[dr] & \\ \text{equivalential} \arrow[dr] & & \begin{array}{c}\text{weakly}\\ \text{algebraizable}\end{array} \arrow[dl] \\ & \text{protoalgebraic} & \end{tikzcd}$$

Figure 1: The main classes in the Leibniz hierarchy.

Given $a^m \in A^\natural$, we set

$$\Box(a^m) \coloneqq \begin{cases} a^m & \text{if } m = 1 \text{ or } m = 2 \\ a^{m-1} & \text{if } m \text{ is even and } m \geq 3 \\ a^{m+1} & \text{if } m \text{ is odd and } m \geq 3. \end{cases}$$

Now, consider the matrix $\langle \boldsymbol{A}^\natural, F^\natural \rangle$, where $F^\natural \coloneqq A_1 \cup A_2$. Observe that the matrix $\langle \boldsymbol{A}^\natural, F^\natural \rangle$ can be constructed out of $\boldsymbol{A}$ in polynomial time, since the arity of the basic operations of $\boldsymbol{A}$ is bounded by 2. The hearth of our proof consists in showing that if $\vdash$ is the logic determined by the matrix $\langle \boldsymbol{A}^\natural, F^\natural \rangle$, then the following conditions are equivalent:

1. $\vdash$ is algebraizable.
2. $\vdash$ is protoalgebraic.
3. h belongs to the clone of $\boldsymbol{A}$.

As a consequence, there is a polynomial time reduction of the $\mathsf{Gen\text{-}Clo}^1_2$ to the problem $\mathsf{Class\text{-}K}$ for every level K of the Leibniz hierarchy. Hence we obtain the following:

Theorem 1. *Let K be a level of the Leibniz hierarchy. $\mathsf{Class\text{-}K}$ is complete for* **EXPTIME**.

References

[1] C. Bergman, D. Juedes, and G. Slutzki. Computational complexity of term-equivalence. *International Journal of Algebra and Computation*, 9(1):113–128, 1999.

[2] J. Czelakowski. *Protoalgebraic logics*, volume 10 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 2001.

[3] J. M. Font. Abstract algebraic logic: An introductory textbook. Forthcoming, 2015.

[4] J. M. Font and R. Jansana. *A general algebraic semantics for sentential logics*, volume 7 of *Lecture Notes in Logic*. A.S.L., 2009. First edition 1996. Electronic version freely available through Project Euclid at https://www.projecteuclid.org/euclid.lnl/1235416965.

[5] R. Freese and M. A. Valeriote. On the complexity of some Maltsev conditions. *International Journal of Algebra and Computation*, 19(1):41–77, 2009.

[6] D. Hobby. Finding type sets is NP-hard. *International Journal of Algebra and Computation*, 1(4):437–444, 1991.

[7] T. Moraschini. A computational glimpse at the Leibniz and Frege hierarchies. Submitted manuscript, January 2016.