Rooted frames for fusions of multimodal logics

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Most of monomodal logics are characterized by classes of frames (see e.g. [1],[2]). It is even possible to use single connected frames for some logics. The additional modalities make the problem of seeking one connected frame more demanding.

Consider a propositional n-modal language $\mathcal{L}_1$ with modal operators $\Box_1, \ldots, \Box_n$ and a propositional m-modal language $\mathcal{L}_2$ with modal operators $\Box_{n+1}, \ldots, \Box_{n+m}$. Let us denote by $\mathcal{L}_{1,2}$ the propositional $n+m$-modal language with operators $\Box_1, \ldots, \Box_n, \Box_{n+1}, \ldots, \Box_{n+m}$. The smallest $n+m$-modal logic in the language $\mathcal{L}_{1,2}$ containing $L_1 \cup L_2$ is called a *fusion* of $L_1 \subset \mathcal{L}_1$ and $L_2 \subset \mathcal{L}_2$. We write $L_1 \oplus L_2$ for the fusion of L_1 and L_2.

A Kripke n-frame $\mathfrak{B} = \langle V, H_1, \ldots, H_n \rangle$ is called a *subframe* of a frame $\mathfrak{F} = \langle W, R_1, \ldots, R_n \rangle$ if $V \subseteq W$ and H_i is the restriction of R_i to V (i.e. $H_i = R_i \cap (V \times V)$), for all $i \in \{1, \ldots, n\}$. A subframe $\mathfrak{B}$ of $\mathfrak{F}$ is called a *generated subframe* of $\mathfrak{F}$ if for each $y \in W$, $y \in V$ if xR_iy for some $x \in V$ and some $i \in \{1, \ldots, n\}$. The subframe of the frame $\mathfrak{F}$ generated by the set $U \subseteq W$ will be denoted by $[U]_{\mathfrak{F}}$. If $U = \{x\}$, we write $[x]_{\mathfrak{F}}$ instead of $[\{x\}]_{\mathfrak{F}}$. For a given class $\mathcal{C}$ of n-frames, let $PGS(\mathcal{C})$ be the class of all subframes of the frames from the class $\mathcal{C}$ generated by a single point. In symbols

$$PGS(\mathcal{C}) = \{[x]_{\mathfrak{F}} \colon \mathfrak{F} = \langle W, R_1, \ldots, R_n \rangle \in \mathcal{C}, x \in W\}.$$

A Kripke n-frame $\mathfrak{F} = \langle W, R_1, \ldots, R_n \rangle$ is *rooted* if $\mathfrak{F} = [x]_{\mathfrak{F}}$ for some $x \in W$ i.e. if there exists $x \in W$ such that for each $y \in W \setminus \{x\}$ there exists a sequence $(x_1, \ldots, x_{k-1})$ of elements from W such that

$$xR_{i_1}x_1, x_1R_{i_2}x_2, \ldots, x_{k-2}R_{i_{k-2}}x_{k-1}, x_{k-1}R_{i_{k-1}}y,$$

where $i_j \in \{1, \ldots, n\}$. The point x is called a *root* of the frame $\mathfrak{F}$.

Let L_1 be an n-modal logic and L_2 be an m-modal logic. Assume that L_1 and L_2 are characterized by classes of rooted frames $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. It is already known that there exists a class of $n+m$-frames that characterizes $n+m$-modal logic $L_1 \oplus L_2$ (see e.g. [3],[4]).

Consider a class $\mathcal{C}$ of rooted frames. Let $\mathfrak{F}$ be a frame with a root x. We say that the point x is a $\mathcal{C}$*-root* if for each $\mathfrak{G} \in \mathcal{C}$ and a root y of $\mathfrak{G}$ there exists a p-morphism from $\mathfrak{F}$ to $\mathfrak{G}$ sending x to y.

Let us consider the class $\mathcal{C}_{Grz.3} = \{\mathfrak{F}^n_{Grz.3} = \langle\{1, \ldots, n\}, \geq\rangle \colon n \in \mathbb{N}\}$ of all finite chains. A frame with a $\mathcal{C}_{Grz.3}$-root is $\mathfrak{F}^r_{Grz.3} = \langle W', \leq\rangle$, where

$$W' = \left\{\frac{1}{n} \colon n \in \mathbb{N}\right\} \cup \{0\}.$$

Let us consider the chain $\mathfrak{F}^6_{Grz.3}$. Point 6 is a root of the frame $\mathfrak{F}^6_{Grz.3}$, therefore $f(0) = 6$. It is necessary to preserve order. In next steps $f(1) = 1$, $f(\frac{1}{2}) = 2$, $f(\frac{1}{3}) = 3$, $f(\frac{1}{5}) = 5$, $f(\frac{1}{k}) = 6$ for $k \geq 6$.

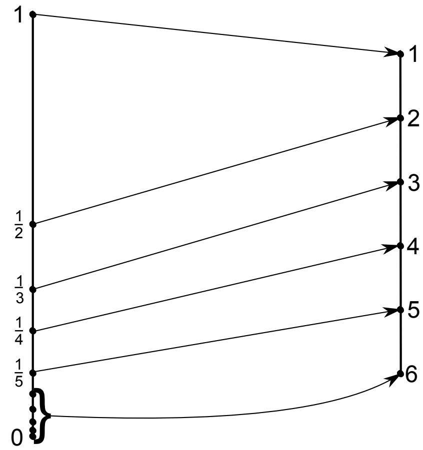

Figure 1: $\mathfrak{F}^r_{Grz.3} \to \mathfrak{F}^6_{Grz.3}$

Let L_1 be an n-modal logic and L_2 be an m-modal logic. Assume that L_1 and L_2 are characterized by classes of rooted frames $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Classes $\mathcal{C}_1'$ and $\mathcal{C}_2'$ are closures of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, under the formation of disjoint unions and isomorphic copies. Moreover, let $\mathfrak{F}^1 = \langle W_1, R_1, \ldots, R_n \rangle$ be an L_1-frame with $PGS(\mathcal{C}_1)$-root and $\mathfrak{F}^2 = \langle W_2, R_{n+1}, \ldots, R_{n+m} \rangle$ be an L_2-frame with $PGS(\mathcal{C}_2)$-root.

In the talk we will show how to construct a rooted frame $\mathfrak{F}^r = \langle W^r, S_1, \ldots, S_{n+m} \rangle$ which characterizes the $n+m$-modal logic $L_1 \oplus L_2$ and has the following properties

(a) $\mathfrak{F}^r$ is countable if $\mathfrak{F}^1$ and $\mathfrak{F}^2$ are countable;

(b) each $S_1, \ldots, S_n$-connected component of the frame $\mathfrak{F}^r$ is isomorphic to the frame $\mathfrak{F}^1$;

(c) each $S_{n+1}, \ldots, S_{n+m}$-connected component of the frame $\mathfrak{F}^r$ is isomorphic to the frame $\mathfrak{F}^2$;

(d) $\mathfrak{F}^r$ is a frame with a $PGS(\mathcal{C}_1' \oplus \mathcal{C}_2')$-root;

(e) for each $n+m$-formula φ, $\mathfrak{F}^r \models \varphi$ if and only if φ is valid in a $PGS(\mathcal{C}_1' \oplus \mathcal{C}_2')$-root of the frame $\mathfrak{F}^r$.

References

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