Semi-Constructive Versions of the Rasiowa-Sikorski Lemma and Possibility Semantics for Intuitionistic Logic

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The celebrated Rasiowa-Sikorski Lemma [6] states that for any Boolean algebra B, any countable set Q of subsets of B, and any non-zero $a \in B$, there exists an ultrafilter U over B such that $a \in U$ and U preserves all existing meets in Q. Rasiowa and Sikorski [5] famously applied the lemma to the Lindenbaum-Tarski algebra of Classical Predicate Logic in order to prove its completeness with respect to Tarskian semantics. However, their proof of the lemma was an application of the Stone representation theorem [7], which relies on the non-constructive Boolean Prime Ideal Theorem (BPI). In fact, Goldblatt [1] observes that the Rasiowa-Sikorski Lemma is equivalent over ZF to the conjunction of BPI and Tarski's Lemma, a weaker proposition that states that for any Boolean algebra B, any countable set Q of subsets of B, and any $a \in B$, there exists an filter F over B such that $a \in F$ and for any $X \in Q$ such that $\bigwedge X$ exists in B, $\bigwedge X \in F$ or there is $x \in X$ such that $\neg x \in F$. Goldblatt also proves that Tarski's Lemma is *semi-constructive*, in the sense that it is equivalent over ZF to the Axiom of Dependent Choices (DC).

In this talk, we provide a generalization of Tarski's Lemma to the variety of distributive lattices (DL), the Q-Lemma, that states that for any distributive lattice L, any countable sets Q_M and Q_J of subsets of L, and any two elements $a, b \in L$, if $a \nleq b$, then there exists a pair (F, I) such that:

- F and I are a filter and an ideal over L respectively;
- $F \cap I = \emptyset$, $a \in F$ and $b \in I$;
- for any $X \in Q_M$, if $\bigwedge X$ exists in L and distributes over all joins, then $\bigwedge X \in F$ or there is $x \in X \cap I$;
- for any $Y \in Q_J$, if $\bigvee Y$ exists in L and distributes over all meets, then $\bigvee Y \in I$ or there is $y \in Y \cap F$.

We show that the Q-Lemma is a semi-constructive version of the Rasiowa-Sikorski Lemma for DL as stated in [2], in the sense that it is equivalent over ZF to Tarski's Lemma, and that the Rasiowa-Sikorski Lemma for DL is equivalent to the conjunction of the Prime Filter Theorem and the Q-Lemma.

Moreover, we generalize some of the ideas behind possibility semantics for classical logic, developed in Holliday [3] and Humberstone [4], to intuitionistic logic. We show in particular how to provide a choice-free bitopological representation of distributive lattices and Heyting algebras based on pairs of filters and ideals, and how this framework combined with the Q-Lemma yields an alternative semantics for Intuitionistic Predicate Logic with the Axiom of Constant Domains.

References

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