Injectivity of (Naturally) Ordered Projection Algebras

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A projection algebra is a set A with an action of the monoid $M = (\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}, \min, \infty)$, where $\mathbb{N}$ is the set of natural numbers, and $n \leq \infty$, for all $n \in \mathbb{N}$. Computer scientists use this notion as a convenient means for algebraic specification of process algebras. In this paper, we study injectivity of ordered projection algebras. We characterize injective cyclic naturally ordered projection algebras as complete posets, also we compare some kinds of weak injectivity such as ideal injectivity and $\mathbb{N}$-injectivity with regular injectivity.

1 Introduction

Algebraic and categorical properties of Projection algebras (or spaces) have been introduced and studied as an algebraic version of ultrametric spaces as well as algebraic structures, for example, in [5, 7, 8, 2]. Computer scientists use this notion as a convenient means for algebraic specification of process algebras (see [6]).

A projection algebra is in fact a set A with an action of the monoid $M = (\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}, \min, \infty)$, where $\mathbb{N}$ is the set of natural numbers, and $n \leq \infty$, for all $n \in \mathbb{N}$. By an *ordered projection algebra* we mean a projection algebra A which is also a poset such that the order is compatible with the action. A *naturally ordered projection algebra* is an ordered projection algebra with the order $a \leq b$ if and only if $a = nb$, for some $n \in \mathbb{N}$. We denote the category of projection algebras by **PRO**, the category of ordered projection algebras by **O-PRO**, and the subcategory of naturally ordered projection algebras by $\mathbf{O\text{-}PRO_{nat}}$.

2 Regular injectivity and Ideal injectivity

In this section, we study injectivity of ordered projection algebras with respect to order emdedding projection maps, so called *regular injectivity*. Also, we compare it with injectivity with respect to embedding of the form $I \to \mathbb{N}^\infty$ for an ideal I of $\mathbb{N}^\infty$, so called *I-injectivity*, and with *ideal injective* which is I-injectivity, for all ideals I of $\mathbb{N}^\infty$.

Theorem 2.1. *Every ordered projection algebras is $\downarrow k$-injective, for $k \in \mathbb{N}$.*

Corollary 2.2. *For ordered projection algebras, ideal injectivity coincides with $\mathbb{N}$-injectivity.*

Theorem 2.3. *For ordered projection algebras, the following are equivalent:*
(1) *Ideal injectivity in* **O-PRO**.
(2) $\mathbb{N}$-*injectivity in* **O-PRO**.
(3) $\mathbb{N}$-*injectivity in* **PRO**.
(4) *Injectivity in* **PRO**.

*Speaker

Theorem 2.4. *A continuously complete naturally ordered projection algebra A is injective in* $\mathbf{O\text{-}PRO_{nat}}$*. But, the converse is not generally true.*

Theorem 2.5. *For a projection algebra A with natural order, the following are equivalent:*

(1) *A is a complete poset.*
(2) *A is a continuously complete ordered projection algebra.*
(3) *A is an infinite countable bounded chain.*
(4) *A is an infinite countable complete chain.*
(5) *A is a cyclic projection algebra.*

Corollary 2.6. *A naturally ordered projection algebra satisfying one of the equivalent conditions of the above theorem is injective in* **PRO***.*

Proposition 2.7. *There is no non trivial regular injective projection algebra with natural order in* **O-PRO***.*

Theorem 2.8 (Baer)**.** *Let A be an ordered projection algebra.*

(1) *If A is injective as an object of* **PRO** *then A is injective with respect to ordered projection algebras with natural order.*

(2) *If A is injective as an object of* **PRO** *with respect to one fixed projection algebras (with natural order) then A is injective with respect to all projection algebras (with natural order).*

(3) *If A is regular injective with respect to embeddings into cyclic* **O-PRO** *then A is regular injective with respect to all projection algebras with natural order.*

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