Duality for Relations on Ordered Algebras

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In algebraic logic, one is accustomed to considering, for example, the inherent order on a distributive lattice as capturing entailment between propositions within a particular logic. Generalizing this to morphisms between algebras, one thinks about binary relations that capture a notion of entailment between logics. At a mimimum, these relations should respect the algebraic structure under consideration, and should in some sense still capture a notion of entailment (that is, order). Respect for the algebraic structure means, essentially, that the relations ought to be relations in the category of the algebras. Capturing entailment means that the relations should be closed under strengthening of premises and weakening of conclusions. Putting these ideas together leads to a natural relational setting for algebraic logic.

Natural duality has its most familiar instances in categories of algebras and spaces that are relevant to (positive) algebraic logic by virtue of being concrete over posets. The objects come equipped with partial order with respect to which the morphisms and operations are monotonic. For example, Priestley duality, Stone duality, Banaschewski duality (between partially ordered sets and Stone distributive lattices), and Hofmann-Mislove-Stralka duality (between semilattices and Stone semilattices) all are concrete over posets. Note that while a Stone space has a trivial order, that fact is precisely the feature that distinguishes a Stone space from a Priestley space. So even Stone duality fits the general ordered scheme, when one takes the duals to be complemented distributive lattices.

We study how one extends a duality between ordered algebras and ordered spaces to relations. The motivation is to understand the general setting in which relation lifting carries over to these dualities.

For this abstract, we restrict our attention only to DL, the category of bounded distributive lattices, and Pri, the category of Priestley spaces. In the full paper we consider a more general setting to include other varieties of algebras and their dual spaces.

In a category $\mathcal{A}$ with pullbacks, one defines $\mathsf{Span}(\mathcal{A})$ as the category of isomorphism classes of spans $A \leftarrow R \rightarrow B$ with composition being defined by pullbacks. So in particular $\mathsf{Span}(\mathsf{DL})$ and $\mathsf{Span}(\mathsf{Pri})$ make sense because both categories have pullbacks.

The categories DL and Pri are both equipped with suitable factorization systems $(\mathcal{E}, \mathcal{M})$ for spans (factoring a span into an epimorphism e followed by a jointly monic span m), so that categories $\mathsf{Rel}(\mathsf{DL})$ and $\mathsf{Rel}(\mathsf{Pri})$ arise by taking morphisms to be the monomorphic spans. In DL, these are essentially sublattices of $A \times B$. In Pri, they are merelty compact subspaces (with the induced order) of the $X \times Y$. Composition is defined by pullback and renormalizing via the factorization system. Again in both $\mathsf{Rel}(\mathsf{DL})$ and $\mathsf{Rel}(\mathsf{Pri})$, this means that composition is concretely the usual relational composition.

Looking toward duality, we are faced immediately with a problem. The dual of a span $A \leftarrow R \rightarrow B$ in distributive lattices is a *cospan* $2^A \rightarrow 2^R \leftarrow 2^B$ in Priestley spaces, and vice versa. Nevertheless, $\mathsf{Rel}(\mathsf{DL})$ provides precisely those relations that respect the algebraic structure of the objects. And $\mathsf{Rel}(\mathsf{Pri})$ provides a similar service for topological structure of Priestley spaces.

To obtain relations that also respect entailment (closure under strengthening of premises and weakening of conclusions), we consider *weakening* relations, i.e., those binary relations between posets that are closed under the following rule: $a \le a'$, $a'Rb'$ and $b' \le b$ implies aRb. Because

DL and Pri are both concrete over Pos, we can define weakening relations between objects to be morphisms in $\mathsf{Rel}(\mathsf{DL})$ or $\mathsf{Rel}(\mathsf{Pri})$ that are closed under the weakening rule.

Putting things together, DL and Pri have suitable structure for defining relations generally, and have forgetful functors into Pos so that weakening relations make (forgetful) sense. Moreover, the composition of relations in DL and Pri coincides concretely with composition of weakening relations in Pos. So we define categories $\overline{\mathsf{DL}}$ and $\overline{\mathsf{Pri}}$ as the subcategories of $\mathsf{Rel}(\mathsf{DL})$ and $\mathsf{Rel}(\mathsf{Pri})$ consisting of relations which forgetfully are weakening relations.

Specifically, in $\overline{\mathsf{DL}}$, a morphism corresponds exactly to a relation closed under the familar proof calculus rules for positive logic. In $\overline{\mathsf{Pri}}$, a morphism is simply a compact upper set in $X^{\mathrm{op}} \times Y$. Notice that these categories are both order-enriched, by taking relations orered by inclusion.

The main problem now is to understand how the natural duality of DL and Pri lifts to $\overline{\mathsf{DL}}$ and $\overline{\mathsf{Pri}}$. Our main additional tool is the weighted limits of cospans and weighted colimits of spans. Call a cospan $P \stackrel{j}{\leftarrow} C \stackrel{k}{\rightarrow} Q$ in Pos *bipartite* if k and j are embeddings and for every $p \in P$, every $q \in Q$, $kq \not\leq jp$. We show that the duals of weakening relations in DL are exactly the bipartite cospans in Pri, and that commas of bipartite cospans in Pri are exact and determine weakening spans in Pri. Thus we have the main theorem.

Theorem 1. *The order enriched categories $\overline{\mathsf{DL}}$ and $\overline{\mathsf{Pri}}$ are dually equivalent on 1-cells and equivalent on 2-cells.*

Now from this duality, we recover the original duality of DL and Pri by noting that is both settings, adjoint pairs of weakening relations determine and are determined by functions. That is, im $\overline{\mathsf{DL}}$, define $\mathsf{Map}(\overline{\mathsf{DL}})$ to consist of pairs of relations (R, S) so that $1_A \leq S \circ R$ and $R \circ S \leq 1_B$. Define $\mathsf{Map}(\overline{\mathsf{Pri}})$ likewise.

Lemma 1. *The category $\mathsf{Map}(\overline{DL})$ is equivalent (actually isomorphic) to DL.*

Now since the duality for relations preserves order on hom-sets, it also follows that $\mathsf{Map}(\overline{Pri})$ is dually equivalent to $\mathsf{Map}(\overline{\mathsf{DL}})$.

Although we have paid attention to Priestley duality here, many of the technical results depend only more general structure of DL and Pri. In the full paper, we discuss sufficient conditions for a natural duality between categories $\mathcal{A}$ and $\mathcal{X}$ that are conrete over Pos to lift to $\overline{\mathcal{A}}$ and $\overline{\mathcal{X}}$.