Undecidability of \{\cdot, 1, \lor\}\-equations in subvarieties of commutative residuated lattices

Gavin St. John$^1$

University of Denver, Denver, Colorado, United States of America
gavin.stjohn@du.edu

The decidability of the equational and quasi-equational theories for commutative residuated lattices ($\text{CRL}$) axiomatized by \{\cdot, 1, \leq\}\-inequalities have been fully classified. It has been shown that quasi-equational theories axiomatized by knotted inequations (k$^m_n$), i.e. universally quantified inequations of the form $x^n \leq x^m$ for $n \neq m$, are not only decidable, but also have the finite embedability property (FEP) \cite{5}. In fact, $\text{CRL} + (k^m_n) + \Gamma$ has the FEP for any set $\Gamma$ of \{\cdot, 1, \leq\}-equations \cite{2}. Viewed proof-theoretically, these results show that the Full Lambek calculus with exchange ($\text{FL}_e$) axiomatized by knotted inference rules have decidable consequence relations.

In \cite{1}, it is shown that $\text{FL}_c$ is undecidable, which algebraically corresponds to $\text{RL} + (k^2_1)$ having undecidable quasi-equational and equational theories. In fact, for $1 \leq n \leq m$, \cite{1} shows that there exists a residuated lattice $R$ in the variety $\text{RL} + (k^m_n)$ such that, for any variety $V$, $R \in V \implies V$ has undecidable quasi-equational and equational theories.

As a consequence of this, certain non-commutative varieties satisfying equations in the signature \{\cdot, 1, \lor\} are also shown to be undecidable.

However, in the commutative case, little is known about the decidability of $\text{CRL}$’s axiomatized by equations in the signature \{\cdot, 1, \lor\}, e.g. the effect of inequations such as $x \leq x^2 \lor 1$ or $xy \leq x^2 y \lor x^3 y^2$ on decidability in $\text{CRL}$ is unknown.

The present work defines a class $D$ of \{\cdot, 1, \lor\}-equations such that the following theorem is obtained:

**Theorem 1.** If (d) $\in D$, then there exists $R_d$ in $\text{CRL} + (d)$ such that for every variety $V$, $R_d \in V \implies V$ has an undecidable quasi-equational theory.

Furthermore, as a consequence of the above theorem, there is a subclass $D' \subset D$ such that

**Corollary 2.** If (d) $\in D'$, then there exists $R_d$ in $\text{CRL} + (d)$ such that for every variety $V$, $R_d \in V \implies V$ has an undecidable equational theory.

As in \cite{1}, \cite{3}, and \cite{4}, we use counter machines (CM), a variant of Turing Machines, for our undecidable problem. From a given a CM $M$ and rule (d) $\in D$, we construct a new machine $M_d$ and a commutative idempotent semi-ring $A_{M_d}$. We interpret machine instructions of $M_d$ as relations on $A_{M_d}$, and define a new relation $<_M$ on $A_N$ such that $M$ halts on input $C \iff \theta(C) <_M q_f$,

where $q_f$ is a designated element $A_{M_d}$ and $\theta$ is a certain function on the configurations of $M$ into the set $A_{M_d}$.
Undecidability of \(\{\cdot, 1, \lor\}\)-equations in subvarieties of commutative residuated lattices

G. St. John

We then simulate the rule (d) in \(A_{M_d}\) by a new relation \(\prec_d\) extending \(\prec_M\) that, on the one hand, satisfies certain restricted consequences of the rule (d), and on the other hand, maintains the property that

\[ M \text{ halts on input } C \iff \theta(C) \prec_d q_f. \]

Lastly, following the methods utilized in [1], we use the theory of residuated frames [2] to construct a residuated lattice \(W^+\) in \(CRL + (d)\) from \(\langle A_{M_d}, \prec_d, \cdot \rangle\) that has the halting problem from the machine \(M\) encoded into the order of \(W^+\), effectively interpreting a halting problem into any variety that contains \(W^+\). Membership in \(D\) is equivalent to whether certain systems of linear equations admit positive solutions. Let

\[
\forall(x_1, \ldots, x_n) \ x_1 x_2 \cdots x_n \leq \bigvee_{(c_1, \ldots, c_n) \in C} x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n}
\]

be the linearization of some \(\{\cdot, 1, \lor\}\)-equation \((r)\), where \(C \subset \mathbb{N}^n\) is finite. Then \((r) \notin D\) if and only if there exists a positive solution to the system of linear equations

\[
\left\{ \sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{n} d_i x_i : (c_1, \ldots, c_n), (d_1, \ldots, d_n) \in C_X \right\},
\]

where

\[ C_X := \{(c_1, \ldots, c_n) \in C : (\exists i \in X) \ c_i > 0\}, \]

for some \(X \subseteq \{1, \ldots, n\}\).

The members of \(D'\) are those equations \((r)\) in \(D\) such that \((r)\) has, as a consequence, an inequation of the form:

\[
(\forall x^n) x^n \leq \bigvee_{i=1}^{m} x^{n+c_i},
\]

where \(n, c_1, \ldots, c_m > 0\), and as a consequence of membership in \(D\), \(m \geq 2\).

References


2