Undecidability of $\{\cdot, 1, \vee\}$-equations in subvarieties of commutative residuated lattices

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The decidability of the equational and quasi-equational theories for commutative residuated lattices ($\mathcal{CRL}$) axiomatized by $\{\cdot, 1, \leq\}$-inequalities have been fully classified. It has been shown that quasi-equational theories axiomatized by *knotted inequations* (k_n^m), i.e. universally quantified inequations of the form $x^n \leq x^m$ for $n \neq m$, are not only decidable, but also have the finite embedability property (FEP) [5]. In fact, $\mathcal{CRL} + (\mathrm{k}_n^m) + \Gamma$ has the FEP for any set Γ of $\{\cdot, 1, \leq\}$-equations [2]. Viewed proof-theoretically, these results show that the Full Lambek calculus with exchange ($\mathbf{FL}_e$) axiomatized by knotted inference rules have decidable consequence relations.

In [1], it is shown that $\mathbf{FL_c}$ is undecidable, which algebraically corresponds to $\mathcal{RL} + (\mathrm{k}_1^2)$ having undecidable quasi-equational and equational theories. In fact, for $1 \leq n \leq m$, [1] shows that there exists a residuated lattice $\mathbf{R}$ in the variety $\mathcal{RL} + (\mathrm{k}_n^m)$ such that, for any variety $\mathcal{V}$,

$$\mathbf{R} \in \mathcal{V} \implies \mathcal{V} \text{ has undecidable quasi-equational and equational theories.}$$

As a consequence of this, certain non-commutative varieties satisfying equations in the signature $\{\cdot, 1, \vee\}$ are also shown to be undecidable.

However, in the commutative case, little is known about the decidability of $\mathcal{CRL}$'s axiomatized by equations in the signature $\{\cdot, 1, \vee\}$, e.g. the effect of inequations such as $x \leq x^2 \vee 1$ or $xy \leq x^2y \vee x^3y^2$ on decidabilty in $\mathcal{CRL}$ is unknown.

The present work defines a class D of $\{\cdot, 1, \vee\}$-equations such that the following theorem is obtained:

Theorem 1. *If* $(\mathrm{d}) \in D$*, then there exists* $\mathbf{R}_\mathrm{d}$ *in* $\mathcal{CRL} + (\mathrm{d})$ *such that for every variety* $\mathcal{V}$*,*

$$\mathbf{R}_\mathrm{d} \in \mathcal{V} \implies \mathcal{V} \textit{ has an undecidable quasi-equational theory.}$$

Furthermore, as a consequence of the above theorem, there is a subclass $D' \subset D$ such that

Corollary 2. *If* $(\mathrm{d}) \in D'$*, then there exists* $\mathbf{R}_\mathrm{d}$ *in* $\mathcal{CRL} + (\mathrm{d})$ *such that for every variety* $\mathcal{V}$*,*

$$\mathbf{R}_\mathrm{d} \in \mathcal{V} \implies \mathcal{V} \textit{ has an undecidable equational theory.}$$

As in [1], [3], and [4], we use *counter machines* (CM), a variant of Turing Machines, for our undecidable problem. From a given a CM M and rule $(\mathrm{d}) \in D$, we construct a new machine M_d and a commutative idempotent semi-ring $\mathbf{A}_{M_d}$. We interpret machine instructions of M_d as relations on $\mathbf{A}_{M_d}$, and define a new relation $<_M$ on $\mathbf{A}_N$ such that

$$M \text{ halts on input } C \iff \theta(C) <_M q_f,$$

where q_f is a designated element A_{M_d} and θ is a certain function on the configurations of M into the set A_{M_d}.

We then simulate the rule (d) in $\mathbf{A}_{M_d}$ by a new relation $<_{\mathbf{d}}$ extending $<_M$ that, on the one hand, satisfies certain restricted consequences of the rule (d), and on the other hand, maintains the property that

$$M \text{ halts on input } C \iff \theta(C) <_{\mathrm{d}} q_f.$$

Lastly, following the methods utilized in [1], we use the theory of residuated frames [2] to construct a residuated lattice $\mathbf{W}^+$ in $\mathcal{CRL}+(\mathrm{d})$ from $\langle A_{M_d}, <_d, \cdot\rangle$ that has the halting problem from the machine M encoded into the order of $\mathbf{W}^+$, effectively interpreting a halting problem into any variety that contains $\mathbf{W}^+$. Membership in D is equivalent to whether certain systems of linear equations admit positive solutions. Let

$$\forall(x_1, ..., x_n)\ x_1 x_2 \cdots x_n \leq \bigvee_{(c_1,\ldots,c_n)\in C} x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n}$$

be the linearization of some $\{\cdot, 1, \vee\}$-equation (r), where $C \subset \mathbb{N}^n$ is finite. Then (r) $\notin D$ if and only if there exists a positive solution to the system of linear equations

$$\left\{ \sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{n} d_i x_i : (c_1, ..., c_n), (d_1, ..., d_n) \in C_X \right\},$$

where

$$C_X := \{(c_1, ..., c_n) \in C : (\exists i \in X)\ c_i > 0\},$$

for some $X \subseteq \{1, ..., n\}$.

The members of D' are those equations (r) in D such that (r) has, as a consequence, an inequation of the form:

$$(\forall x)\ x^n \leq \bigvee_{i=1}^{m} x^{n+c_i},$$

where $n, c_1, ..., c_m > 0$, and as a consequence of membership in D, $m \geq 2$.

References

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