A Constructive Four-Valued Logic

Yuanlei Lin and Minghui Ma

Institute of Logic and Cognition, Sun Yat-Sen University, Guangzhou, China
linyuanlei@126.com,mamh6@mail.sysu.edu.cn

1 Introduction

The Belnap-Dunn four-valued logic is the logic of De Morgan lattices. A De Morgan lattice is an algebra $(A, \wedge, \vee, \neg, 0, 1)$ where $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\neg$ is the De Morgan negation, namely $\neg$ is an unary operation on A satisfying the following conditions:

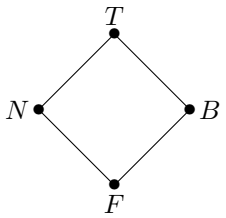
(1) $\neg(a \wedge b) = \neg a \vee \neg b$;

(2) $\neg(a \vee b) = \neg a \wedge \neg b$;

(3) $\neg\neg a = a$;

(4) $\neg 0 = 1$ and $\neg 1 = 0$.

It is well-known that every De Morgan lattice can be embedded in a (subdirect) product of the lattice **4**. Belnap's four-valued logic [1] is the logic of the following lattice **4**:



Belnap's Lattice **4**

Dunn's logic of De Morgan lattices [4, 5] is the same as Belnap's four-valued logic due to the representation theorem. Dunn [2] developed a theory of negation which is adapted with the theory of information. For a comprehensive survey on negation, see Dunn [4].

In the present paper, we shall present a constructive four-valued logic **C4L**. Dunn's four-valued semantics for De Morgan logic introduces two semantics consequence relations $\varphi \models_1 \psi$ and $\varphi \models_0 \psi$ which can be interpreted via Belnap's concepts of acceptance and rejection. A formula φ is accepted if 1 belongs to the value of φ, and it is rejected if 0 belongs to the value of φ. Our idea is to generalize Belnap-Dunn four-valued logic to a weak logic which is constructive in the following sense: if a formula is accepted, then it is accepted at any future state, and if it is rejected, it is rejected at any future state. The underling temporal structure is assumed to be a linear order. We call this logic as a constructive four-valued logic because it is a sublogic of Belnap-Dunn four-valued logic in which the law of double negation elimination is refuted. The logic **C4L** can be viewed the weakening of Belnap-Dunn logic in the way that intuitionistic logic is the weakening of classical propositional logic. We shall introduce the Kripke semantics for **C4L**. Consequently, Belnap-Dunn four-valued logic can be represented as the logic of a single reflexive point which is an extension of **C4L**.

The negation in **C4L** is a new one to Dunn's kite of negations [4]. Intuitively it is a modal negation which is interpreted on linearly ordered sets.

2 Language and Semantics

The language $\mathcal{L}$ of the constructive four-valued logic **C4L** consists of a denumerable set of propositional variables $\mathbf{Var} = \{p_i \mid i \in \mathbb{N}\}$, and propositional connectives $\top, \bot, \neg, \wedge$ and $\vee$. The set of all formulae is defined by the following inductive rule:

$$\varphi ::= p \mid \top \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi, \text{ where } p \in \mathbf{Var}.$$

A *sequent* is an expression of the form $\varphi \vdash \psi$ where φ and ψ are formulae.

Definition 2.1. A *frame* is a pair $\mathcal{F} = (W, \leq)$ where W is a non-empty set of states, and $\leq$ is a linear order on W. A subset $A \subseteq W$ is called an *upset* in $\mathcal{F}$ if A is closed under $\leq$, namely, if $w \in A$ and $w \leq w'$, then $w' \in A$. Let $Up(W)$ be the set of all upsets in $\mathcal{F}$.

A *model* is a tuple $\mathcal{M} = (W, \leq, V)$, where $(W, \leq)$ is a frame, and $V : \mathbf{Var} \to Up(W) \times Up(W)$ is valuation. When $V(p) = (A^+, A^-)$, we say that A^+ is the set of states which accept p, and A^- is the set of states which reject p. We also write $V^+(p) = A^+$ and $V^-(p) = A^-$.

Obviously for any frame $\mathcal{F} = (W, \leq)$, we have $\emptyset, W \in Up(W)$. Moreover, in any model $\mathcal{M} = (W, \leq, V)$, each propositional variable p is assigned with a pair $(V^+(p), V^-(p))$ of upsets. There is no need to require that $V^+(p) \cup V^-(p) = W$ because there may be states which neither accept nor reject a propositional variable.

Definition 2.2. For any formula φ, model $\mathcal{M} = (W, \leq, V)$ and $w \in W$, the acceptance and rejection relations $\mathcal{M}, w \models^+ \varphi$ and $\mathcal{M}, w \models^- \varphi$ are defined inductively as follows:

(1) $\mathcal{M}, w \models^+ \top$ and $\mathcal{M}, w \not\models^- \top$; $\mathcal{M}, w \not\models^+ \bot$ and $\mathcal{M}, w \models^- \bot$.

(2) $\mathcal{M}, w \models^+ p$ iff $w \in V^+(p)$; $\mathcal{M}, w \models^- p$ iff $w \in V^-(p)$.

(3) $\mathcal{M}, w \models^+ \varphi \wedge \psi$ iff $\mathcal{M}, w \models^+ \varphi$ and $\mathcal{M}, w \models^+ \varphi$.

(4) $\mathcal{M}, w \models^- \varphi \wedge \psi$ iff $\mathcal{M}, w \models^- \varphi$ or $\mathcal{M}, w \models^- \psi$.

(5) $\mathcal{M}, w \models^+ \varphi \vee \psi$ iff $\mathcal{M}, w \models^+ \varphi$ or $\mathcal{M}, w \models^+ \psi$.

(6) $\mathcal{M}, w \models^- \varphi \vee \psi$ iff $\mathcal{M}, w \models^- \varphi$ and $\mathcal{M}, w \models^- \psi$.

(7) $\mathcal{M}, w \models^+ \neg\varphi$ iff for any $w' \in W$, if $w \leq w'$, then $\mathcal{M}, w' \models^- \varphi$.

(8) $\mathcal{M}, w \models^- \neg\varphi$ iff there is $w' \in W$ such that $w \leq w'$ and $\mathcal{M}, w' \models^+ \varphi$.

A formula φ is *accepted in* $\mathcal{M}$, notation $\mathcal{M} \models^+ \varphi$, if for any $w \in W$, $\mathcal{M}, w \models^+ \varphi$. A formula ψ is a *positive consequence* of φ, notation $\varphi \models^+ \psi$, if for any model $\mathcal{M}$ and w in $\mathcal{M}$, if $\mathcal{M}, w \models^+ \varphi$, then $\mathcal{M}, w \models^+ \psi$. We say that ψ is a *negative consequence* of φ, notation $\varphi \models^- \psi$, if for any model $\mathcal{M}$ and w in $\mathcal{M}$, if $\mathcal{M}, w \models^- \varphi$, then $\mathcal{M}, w \models^- \psi$.

Lemma 2.3. For any model $\mathcal{M} = (W, \leq, V)$ and $w, w' \in W$, the following hold:

(1) If $w \leq w'$ and $\mathcal{M}, w \models^+ \varphi$ then $\mathcal{M}, w' \models^+ \varphi$.

(2) If $w \leq w'$ and $\mathcal{M}, w \models^- \varphi$ then $\mathcal{M}, w' \models^- \varphi$.

(3) $\mathcal{M}, w \models^- \varphi$ iff $\mathcal{M}, w \models^+ \neg\varphi$.

(4) $\mathcal{M} \models^- \varphi$ iff $\mathcal{M} \models^+ \neg\varphi$.

Corollary 2.4. *For any formulae φ and ψ, $\varphi \models^- \psi$ iff $\neg\varphi \models^+ \neg\psi$.*

We say that a sequent $\varphi \vdash \psi$ is *valid* if $\varphi \models^+ \psi$ and $\psi \models^- \varphi$. A sequent rule of the form

$$\frac{\varphi_1 \vdash \psi_1, \ldots, \varphi_n \vdash \psi_n}{\varphi_0 \vdash \psi_0}(\rho)$$

is *valid*, if the premisses $\varphi_i \vdash \psi_i$ for $1 \leq i \leq n$ are valid, then $\varphi_0 \vdash \psi_0$ is valid.

Proposition 2.5. *The following sequents and rule are valid:*

(1) $\neg\neg\neg\neg\varphi \vdash \neg\neg\varphi$.

(2) $\neg(\varphi \wedge \psi) \vdash \neg\varphi \vee \neg\psi$ *and* $\neg\varphi \vee \neg\psi \vdash \neg(\varphi \wedge \psi)$.

(3) $\neg(\varphi \vee \psi) \vdash \neg\varphi \wedge \neg\psi$ *and* $\neg\varphi \wedge \neg\psi \vdash \neg(\varphi \vee \psi)$.

(4) $\varphi \wedge \neg\neg\psi \vdash \neg\neg\varphi \vee \psi$.

(5) *The contraposition rule:*

$$\frac{\varphi \vdash \psi}{\neg\psi \vdash \neg\varphi}(CP)$$

One can easily show that the sequents $\neg\neg\varphi \vdash \varphi$ and $\neg\neg\neg\varphi \vdash \neg\varphi$ are not valid.

3 The Logic C4L

The set of all valid sequents under the Kripke semantics can be axiomatized as a sequent system. We have the following definition of the constructive logic **C4L**:

Definition 3.1. *The logic* **C4L** *consists of the following axioms and rules:*

(1) *Axioms:*

$$\text{(Id) } \varphi \vdash \varphi \quad (\bot)\ \bot \vdash \varphi \quad (\top)\ \varphi \vdash \top \quad \text{(D) } \varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$$

$$(\neg\bot)\ \varphi \vdash \neg\bot \quad (\neg\top)\ \neg\top \vdash \varphi \quad \text{(N1) } \neg\varphi \vdash \neg\neg\neg\varphi \quad \text{(N2) } \varphi \wedge \neg\neg\psi \vdash \neg\neg\varphi \vee \psi$$

$$\text{(DM1) } \neg(\varphi \vee \psi) \vdash \neg\varphi \wedge \neg\psi \quad \text{(DM2) } \neg\varphi \wedge \neg\psi \vdash \neg(\varphi \vee \psi)$$

(2) *Rules for lattice operations:*

$$\frac{\varphi_i \vdash \psi}{\varphi_1 \wedge \varphi_2 \vdash \psi}\ (\wedge\text{L})(i = 1, 2) \quad \frac{\varphi \vdash \psi \quad \varphi \vdash \chi}{\varphi \vdash \psi \wedge \chi}\ (\wedge\text{R})$$

$$\frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}\ (\vee\text{L}) \quad \frac{\varphi \vdash \psi_i}{\varphi \vdash \psi_1 \vee \psi_2}\ (\vee\text{R})(i = 1, 2)$$

(3) *Cut rule and contraposition rule:*

$$\frac{\varphi \vdash \psi \quad \psi \vdash \chi}{\varphi \vdash \chi}\ (\text{Cut}) \quad \frac{\varphi \vdash \psi}{\neg\psi \vdash \neg\varphi}\ (\text{CP})$$

A sequent $\varphi \vdash \psi$ is derivable in **C4L** *if there is a derivation of the sequent.*

Theorem 3.2. *A sequent $\varphi \vdash \psi$ is derivable in* **C4L** *if and only if $\varphi \vdash \psi$ is valid.*

The logic **C4L** is a sublogic of Belnap-Dunn four-valued logic. From the semantic perspective, if one considers the special model which has only one single reflexive state $\circ$, then the set of all valid sequents in this frame is exactly the Belnap-Dunn four-valued logic.

4 Weak De Morgan Algebras

From algebraic perspective, there is a class of algebras for **C4L**. We call them *weak De Morgan algebras*. We shall prove the algebraic completeness of **C4L**, and develop a cut-free sequent calculus for **C4L** by which we get the decidability of the derivation of a sequent in the system.

Definition 4.1. A *weak De Morgan algebra* is an algebra $(A, \wedge, \vee, \neg, 0, 1)$ where $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $\neg$ is an operation on A satisfying the following conditions:

(1) $\neg(a \wedge b) = \neg a \vee \neg b$.

(2) $\neg(a \vee b) = \neg a \wedge \neg b$.

(3) $\neg a \leq \neg\neg\neg a$.

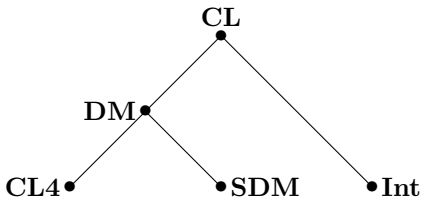
(4) $a \wedge \neg\neg\neg b \leq \neg\neg a \vee \neg b$.

(5) $\neg 0 = 1$ and $\neg 1 = 0$.

The class of all weak De Morgan algebras is denoted by **WDM**.

Theorem 4.2. *A sequent $\varphi \vdash \psi$ is derivable in* **C4L** *iff $\varphi \vdash \psi$ is valid in* **WDM**.

Obviously the class of all De Morgan algebras is a subvariety of **WDM**. Note that the difference between De Morgan algebras and weak De Morgan algebras is about number of negations. Semi-De Morgan algebras proposed in [6] are weakened from De Morgan algebras by dropping the double negation law and one De Morgan law. However, the logic of semi-De Morgan algebras (**SDM**) is incomparable with **C4L**. The following map of logics show their relationships with classical and intuitionistic logic:



Gentzen sequent calculus for **C4L** shall be presented in the full version of this abstract.

References

[1] N. Belnap. A useful four-valued logic. In: J. M. Dunn and G. Epstein (eds.), *Modern Uses of Multiple-Valued Logic*, pp. 5–37, 1977.

[2] M. Dunn. *The Algebra of Intensional Logics*. Ph.D. Dissertation, University of Pittsburg, 1966.

[3] M. Dunn. A kripke-style semantics for first-degree relevant implications (abstract). *The Journal of Symbolic Logic* 36, 362–363, 1971.

[4] M. Dunn. A comparative study of various model-theoretic treatments of negation: a history of formal negation. In: D. M. Gabbay and H. Wansing (eds.), *What is Negation?*, pp. 23–51. Kluwer Academic Publishers, 1999.

[5] M. Dunn. Partiality and its dual. *Studia Logica*, 65: 5-40, 2000.

[6] Hanamantagouda P. Sankappanavar. Semi-De Morgan algebras. *The Journal of Symbolic Logic*, 52(3): 712–724, 1987.