On the Complexity of the Equational Theory of Generalized Residuated Boolean Algebras

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1 Introduction

A residuated Boolean algebra is an algebra $(A, \wedge, \vee, ', \top, \bot, \bullet, \backslash, /)$ where $(A, \wedge, \vee, ', \top, \bot)$ is a Boolean algebra, and $\bullet$, $\backslash$ and $/$ are binary operators on A satisfying the following residuation property: for any $a, b, c \in A$,

$$a \bullet b \leq c \text{ iff } b \leq a \backslash c \text{ iff } a \leq c/b$$

The operators $\backslash$ and $/$ are called right and left residuals of the fusion $\bullet$ respectively.

Residuated boolean algebras are introduced by Jónsson and Tsinakis [3] as generalizations of relation algebras. Jispen [2] proved that the equational theory of residuated boolean algebras with unit, and that of many relative classes of algebras are decidable. Buszkowski [1] showed the finite embeddability property for residuated boolean algebras, which yields the decidability of the universal theory of residuated boolean algebras. The complexity of the equational theory of residuated boolean algebras is solved in [4], where the main result is that the equational theory of residuated boolean algebras is PSPACE-complete.

Generalized residuated Boolean algebras are introduced in [1]. The generalization is from binary to arbitrary $n \geq 2$ ary residuals. Instead of a single binary operator $\bullet$, generalized residuated algebras admit a finite number of finitary operations o. With each n-ary operation o_i $(1 \leq i \leq m)$ there are associated n residual operations o/j $(1 \leq j \leq n)$ which satisfy the following generalized residuation law:

$$o_i(a_1, \ldots, a_n) \leq b \quad \text{iff} \quad a_j \leq (o_i/j)(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n)$$

A *generalized residuated Boolean algebra* is a Boolean algebra with generalized residual operations. The aim of this paper is to show that the complexity of the equational theory of such algebras is still PSAPCE-complete. Our proof is by reducing the decidability of the equational theory into the decidability of a sequent calculus for generalized Boolean residuated algebra.

2 Generalized BFNL

The sequent calculus for Boolean residuated algebras, namely Boolean full nonassociative Lambek calculus (BFNL), is introduced in [1]. Here we shall introduce the sequent calculus GBFNL for generalized residuated Boolean algebras. The formulae are defined as usual (cf. [4]). Structures are defined inductively as follows:

(1) All formulae are structures.

(2) For n-ary operator o_i $(n \geq 2)$ and structures $\Gamma_1, \ldots, \Gamma_n$, $(\Gamma_1, \ldots, \Gamma_n)_{o_i}$ is a structure.

By $\Gamma[\,]$ we mean a structure with a single position which can be filled with a structure.

Definition 2.1. *The sequent calculus GBFNL for generalized residuated Boolean algebras consists of the following axioms and rules:*

(1) Axioms:

$$(Id)\ A \Rightarrow A \quad (D)\ A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$$

$$(\top)\ \Gamma \Rightarrow \top \quad (\bot)\ \Gamma[\bot] \Rightarrow A \quad (\neg 1)\ A \wedge \neg A \Rightarrow \bot \quad (\neg 2)\ \top \Rightarrow A \vee \neg A$$

(2) Rules:

$$\frac{\Gamma[(A_1, \ldots, A_n)_{o_i}] \Rightarrow A}{\Gamma[o_i(A_1, \ldots, A_n)] \Rightarrow A}(\mathrm{o_iL}) \quad \frac{\Gamma_1 \Rightarrow A_1; \ldots; \Gamma_n \Rightarrow A_n}{(\Gamma_1, \ldots, \Gamma_n)_{o_i} \Rightarrow o_i(A_1, \ldots, A_n)}(\mathrm{o_iR})$$

$$\frac{\Delta[A_j] \Rightarrow B; \Gamma_1 \Rightarrow A_1; \ldots; \Gamma_n \Rightarrow A_n}{\Delta[(\Gamma_1, \ldots, (o_i/j)(A_1, \ldots, A_n), \ldots, \Gamma_n)_{o_i}] \Rightarrow B}(\mathrm{o_i/jL})$$

$$\frac{(A_1, \ldots, \Gamma, \ldots, A_n)_{o_i} \Rightarrow A_j}{\Gamma \Rightarrow o_i/j(A_1, \ldots, A_n)}(\mathrm{o_i/jR})$$

$$\frac{\Gamma[A_i] \Rightarrow B}{\Gamma[A_1 \wedge A_2]}(\wedge \mathrm{L}) \quad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}(\wedge \mathrm{R})$$

$$\frac{\Gamma[A_1] \Rightarrow B \quad \Gamma[A_2] \Rightarrow B}{\Gamma[A_1 \vee A_2] \Rightarrow B}(\vee \mathrm{L}) \quad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}(\vee \mathrm{R}) \quad \frac{\Delta \Rightarrow A \quad \Gamma[A] \Rightarrow B}{\Gamma[\Delta] \Rightarrow B}(\mathrm{Cut})$$

The sequent calculus GBFNL can be simulated by a multi-sorted Boolean nonassociative Lambek calculus which is denoted by MBFNL. This means that n-ary residuals can be translated into binary ones. A translation ‡ from GBFNL to MBFNL can be defined inductively as usual. In particular, we have the following translation of residuals:

(1) $((A_1, \ldots, A_n)_{o_i})^{\ddagger} = (\cdots((A_1 \bullet_i A_2) \cdots) \bullet_i A_n)$.

(2) $(o_i/j)(A_1, \ldots, A_n)^{\ddagger} = (\cdots(A_1 \bullet_i A_2) \cdots) \bullet_i A_{j-1}) \backslash_i (\cdots(A_j /_i A_n) \cdots) /_i A_{j+1})$.

(3) $(\Gamma_1, \ldots, \Gamma_n)^{\ddagger}_{o_i} = (\cdots((\Gamma_1 \circ_i \Gamma_2) \cdots) \circ_i \Gamma_n)$.

This translation is faithful. We may easily obtain the following theorem of simulation:

Theorem 2.2. $\vdash_{\mathrm{GBNFL}} \Gamma \Rightarrow A$ *iff* $\vdash_{\mathrm{MBFNL}} \Gamma^{\dagger} \Rightarrow A^{\dagger}$.

3 Complexity of GBFNL

The second step to solve the complexity problem is to simulate MBFNL by a multi-sorted tense logic MKt which is the multi-modal version of basic tense logic Kt. The translation # defined in [4], which embeds BFNL into two-sorted tense logic $\mathrm{K}^{t}_{1,2}$, can be extended to simulate MBFNL. Each n-ary product operator o_i is translated via n pairs of tense operators. The following results can be obtained as in [4].

Theorem 3.1. $\vdash_{\mathrm{MBFNL}} \Gamma \Rightarrow A$ *iff* $\vdash_{\mathrm{MKt}} (f(\Gamma))^{\#} \supset A^{\#}$.

Moreover, using the technique in [4], one can simulate MKt by the basic tense logic Kt via a similar translation $*$ as in [4].

Theorem 3.2. $\vdash_{\mathrm{MKt}} A$ *iff* $\vdash_{\mathrm{Kt}} A^{*}$.

Since the complexity of Kt is PSPACE-complete, it follows that GBFNL is in PSPACE. On the other hand, we may define a translation † from the modal logic K to GBNFL as in [4] such that $(\Diamond A)^\dagger = o(m_1, \ldots, m_{n-1}, A)$. Then we obtain the following simulation result:

Theorem 3.3. *For any modal formula A, $\vdash_{\mathrm{K}} A$ iff $\vdash_{\mathrm{GBNFL}} \top \Rightarrow A^\dagger$.*

Since the modal logic K is PSAPCE-complete, it follows that GBFNL is PSPACE-hard. Therefore we get the following theorem:

Theorem 3.4. *GBFNL is PSPACE-complete.*

As a consequence, the equational theory of generalized Boolean residuated algebras is PSPACE-complete.

4 More complexity results

If we change the Boolean basis of a generalized Boolean residuated algebra into distributive lattices, we get generalized distributive residuated lattices. We also obtain the generalized distributive full nonassociative Lambek calculus (GDFNL) for such algebras.

Theorem 4.1. *GBFNL is a conservative extension of GDFNL.*

It follows that GDFNL is in PSPACE. We reduce the satisfiability of a QBF to the validity of consequence relation of distributive lattice with bi-tense operators. The equational theory of distributive lattices with bi-tense operators is PSPACE-hard. Then GDFNL is PSPACE-hard.

Theorem 4.2. *GDFNL is PSPACE-complete. Hence DFNL is PSPACE-complete.*

References

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