Deepening the link between logic and functional analysis via Riesz MV-algebras

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Riesz Spaces are lattice-ordered linear spaces over the field of real numbers $\mathbb{R}$. They have had a predominant rôle in the development of functional analysis over ordered structures, due to the simple remark that most of the spaces of functions one can think of are indeed Riesz Spaces.

Not very known is the rôle that vector lattices play in logic. Given any positive element u of a Riesz Space V, the interval $[0, u]$ can be endowed with a stucture of Riesz MV-algebra. These structures have been defined in the setting of Łukasiewicz logic, as expansion of MV-algebras – the *standard* semantics of the infinite valued Łukasiewicz logic – and in [1] is proved that Riesz MV-algebras are categorical equivalent to Riesz Spaces with a strong unit. Henceforth, vector lattices and logic are closely related.

Our aim is to exploit the connection between Riesz Spaces and MV-algebras to deepen the link between functional analysis and Łukasiewicz logic.

The first step is to introduce a notion of *limit of formulas* and use it to characterize the (uniform) norm convergence in Riesz Spaces. Consider the logic $\mathbb{R}\mathcal{L}$ that has Riesz MV-algebras as models (and it is a conservative expansion of Łukasiewicz logic) and let η_r denote the formula $\Delta_r\top$ of $\mathbb{R}\mathcal{L}$, where $\{\Delta_r\}$ is the family of connectives that models the scalar operation and $\top$ is defined as usual. Thus, $\Delta_r\top$ is evaluated into r by any $[0, 1]$-evaluation.

Definition 1. *We say that φ is the limit of the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$, and we write $\varphi = \lim_n \varphi_n$ if for any $r \in [0, 1)$ there exists k such that $\vdash \eta_r \to (\varphi \leftrightarrow \varphi_n)$ for any $n \geq k$.*

This notion of *logical limit* is strictly connected to the one of convergence. Indeed, it is well known that the Lindenbaum-Tarski algebra of Łukasiewicz logic is isomorphic with the algebra of piecewise linear functions with integer coefficients. The same holds for the Lindenbaum-Tarski algebra of $\mathbb{R}\mathcal{L}$, which is isomorphic with the algebra of piecewise linear functions with real coefficients. If we denote by f_φ the function that correspond to the equivalence class of φ in the Lindenbaum-Tarski algebra of $\mathbb{R}\mathcal{L}$, we have the following result.

Theorem 1. *The following are equivalent*
(1) $\lim_n \varphi_n = \varphi$,
(2) $\lim_n f_{\varphi_n} = f_\varphi$ *(uniform convergence in the Lindenbaum-Tarski algebra of* $\mathbb{R}\mathcal{L}$*),*
(3) $f_{\varphi_n} \to f_\varphi$ *(order convergence in the Lindenbaum-Tarski algebra of* $\mathbb{R}\mathcal{L}$*).*

The above-mentioned result allows us to explore the possibility of studying the norm-completion of the Lindenbaum-Tarski algebra of $\mathbb{R}\mathcal{L}$ by its Dedekind σ-completion.

If RL_n denotes the Lindenbaum-Tarski algebra of $\mathbb{R}\mathcal{L}$ (where formulas have at most n variables), we characterize two different norm-completions of it. To do so, we consider the following notations:
(i) $\|[\varphi]\|_u = \sup\{f_\varphi(\mathbf{x}) | \mathbf{x} \in [0, 1]^n\}$, where $[\varphi] \in RL_n$,
(ii) $\|f\|_\infty = \sup\{f(\mathbf{x}) | \mathbf{x} \in [0, 1]^n\}$, where $f \in C([0, 1]^n)$,

(iii) $I([\varphi]) = \int f_\varphi(\mathbf{x})d\mathbf{x}$, where $[\varphi] \in RL_n$.
All of the above defined operator are norms in the corresponding spaces and the following theorem holds.

Theorem 2. *(1) The norm-completion of the normed space $(RL_n, \|\cdot\|_u)$ is isometrically isomorphic with $(C([0,1]^n), \|\cdot\|_\infty)$,*

(2) The norm-completion of the normed space (RL_n, I) is isometrically isomorphic with $(L^1(\mu)_u, s_\mu)$, where

(i) μ be the Lebesgue measure associated to I,

(ii) $L^1(\mu)_u$ is the algebra of $[0,1]$-valued integrable functions on $[0,1]^n$,

(iii) $s_\mu(\hat{f}) = I(f)$ and $\hat{f}$ is the class of f, provided we identify two functions that are equal μ-almost everywhere.

With the goal of capturing the unit norm $\|\cdot\|_u$ in a purely syntactic way, we now define the logical system $\mathcal{IRL}$, which stands for *Infinitary Riesz Logic*, by adding an infinitary disjunction to the systems $\mathbb{RL}$ (as well as appropriate axioms and a deduction rule). The system has Dedekind σ-complete Riesz MV-algebras as models and the following results hold.

Theorem 3. *(1) IRL, the Lindenbaum-Tarski algebra of $\mathcal{IRL}$ is a Dedekind σ-complete Riesz MV-algebra;*

(2) $\mathcal{IRL}$ is complete with respect to to all algebras in $\mathbf{RMV_{dc\sigma}}$, the class of Dedekind σ-complete Riesz MV-algebras;

Moreover, we can characterize the models of $\mathcal{IRL}$ by means of particular compact Hausdorff spaces.

Theorem 4. *(1) All Dedekind σ-complete Riesz MV-algebras are norm-complete w.r.t. $\|\cdot\|_u$.*

(2) Let A be a Dedekind σ-complete Riesz MV-algebra. There exists a quasi-Stonean compact Hausdorff space (i.e. it has a base of open F_σ sets) X such that $A \simeq C(X)_u$, the unit interval of $C(X)$.

We conclude this abstract by recalling how all the different completions of RL_n we have defined are linked to each other.

(1) If one consider the sup-norm, the norm completion of RL_n is $C([0,1]^n, \|\cdot\|_\infty)$, which is not Dedekind complete and it is contained in IRL_n.

(2) IRL is a norm-complete Riesz MV-algebra and IRL_n contains $C([0,1]^n)$, as the latter is the norm-completion of RL_n.

(3) If one consider the integral norm, the norm completion of RL_n is $L^1(\mu)_u$, i.e. the unit interval of the space of μ-integrable functions in n variables. This space is Dedekind complete as it is an abstract L-space and hence contains IRL_n.

References

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