Compact Enriched Categories

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The interplay between order and topology has attracted a great deal of attention of researchers working in these fields. Of particular inspiration to us is the work [7] of Nachbin about topological spaces equipped with an additional partial order relation, subject to certain compatibility conditions. A particular class of such spaces, the compact ones, can be equivalently described in purely topological terms: the category of partially ordered compact spaces and monotone continuous maps is equivalent to the category of stably compact spaces and spectral maps (see [2] for details). As a consequence, the scope of various important notions and results in topology can be substantially extended, we mention here the concept of order-normality and the Urysohn Lemma. Turning the emphasis "up-side down", one might also ask what properties of the partial order are guaranteed by the existence of a compatible compact topology? One quick answer to this question is implied by [5, Lemma II.1.9]: since every partially ordered compact space corresponds to a stably compact space which is in particular sober, every partially ordered compact space has directed suprema.

Another important source of inspiration for our research over the past years has been Lawvere's ground-breaking paper [6] presenting generalised metric spaces as "order relations enriched in the quantale $[0, \infty]$", or better: as enriched categories. Undoubtedly, topology is omnipresent in the study of metric spaces; however, there does not seem to exist a systematic account in the literature connecting both lines of research.

The principal aim of this talk is to investigate compact quantale-enriched categories, encompassing this way ordered, metric, and probabilistic metric compact spaces. We place this study in the general framework of topological theories [3] and monad-quantale-enriched categories (see [4]).

Accordingly, in this talk we consider an (almost) strict topological theory $\mathcal{U} = (\mathbb{U}, \mathcal{V}, \xi)$ in the sense of [3] based on the ultrafilter monad $\mathbb{U}$, on a quantale $\mathcal{V}$ and on a convergence relation $\xi : U\mathcal{V} \to \mathcal{V}$ that makes $\mathcal{V}$ a compact Hausdorff topological space. The term "almost strict" refers to the fact that we do not require continuity of $\otimes$ but only lax continuity. Based on this data, one obtains a natural extension of the ultrafilter monad on Set to a monad (U, m, e) on $\mathcal{V}$-Cat [8] such that $e_X : X \to UX$ and $m_X : UUX \to UX$ become $\mathcal{V}$-functors, for each $\mathcal{V}$-category (X, a_0). The objects of the Eilenberg-Moore category for this monad, $\mathcal{V}\text{-}\mathsf{Cat}^{\mathbb{U}}$, can be described as triples (X, a_0, α) where (X, a_0) is a $\mathcal{V}$-category and $\alpha : UX \to X$ is the convergence of a compact Hausdorff topology on X. For $\mathcal{V}$ being the two-element lattice with the discrete topology, compact $\mathcal{V}$-categories coincide with Nachbin's ordered compact Hausdorff spaces; correspondingly, for $\mathcal{V}$ being Lawvere's quantale $[0, \infty]$ with the canonical compact Hausdorff topology, we call compact $\mathcal{V}$-categories *metric compact Hausdorff spaces*.

We recall that ordered compact Hausdorff spaces can be considered as special topological spaces, via a comparison functor

$$\begin{tikzcd} \mathsf{Ord}^{\mathbb{U}} \arrow[r] \arrow[dr] & \mathsf{Top} \arrow[d] \\ & \mathsf{Ord} \end{tikzcd}$$

commuting with the forgetful functors to the category Ord. To bring this construction into our setting, we consider the category $\mathcal{U}$-Cat of $\mathcal{U}$-categories and $\mathcal{U}$-functors. A $\mathcal{U}$-category is a pair (X, a) were a is a $\mathcal{U}$-relation, meaning that it is a V-relation of the type $UX \nrightarrow X$, subject to reflexivity and transitivity. Furthermore, $\mathcal{U}$-relations compose through Kleisli convolution, $\circ$; unfortunately, associativity of this operation depends on the continuity of $\otimes$. Due to this fact, extreme care is needed when handling notions and results transferred from the framework of $\mathcal{V}$-categories. In this talk we revise and expand results regarding "Lawvere completeness in topology" (see [1]). For instance, since in general $\mathcal{U}$-distributors do not compose, the notion of Lawvere-complete $\mathcal{U}$-category is defined with respect to the set of those adjunctions $\varphi \dashv \psi$ where the composites $\varphi \circ \psi$ and $\psi \circ \varphi$ are $\mathcal{U}$-distributors.

Finally, there are suitable functors $(-)_0 : \mathcal{U}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Cat}$ and $K : (\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{U}} \to \mathcal{U}\text{-}\mathsf{Cat}$ that make the diagram

$$\begin{tikzcd} (\mathcal{V}\text{-}\mathsf{Cat})^{\mathbb{U}} \arrow[r, "K"] \arrow[dr, "G^{\mathbb{U}}"'] & \mathcal{U}\text{-}\mathsf{Cat} \arrow[d, "(-)_0"] \\ & \mathcal{V}\text{-}\mathsf{Cat} \end{tikzcd}$$

commutative. In particular, under some conditions that also include the continuity of the tensor $\otimes$, we have proven that compact Hausdorff $\mathcal{V}$-categories are Lawvere-complete since compact Hausdorff $\mathcal{V}$-categories correspond to Lawvere-complete $\mathcal{U}$-categories and the functor $(-)_0$ preserves Lawvere-completeness.

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