Tensor products of Cuntz semigroups

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Abstract Cuntz semigroups, also called Cu-semigroups, are domains with a compatible semigroup structure. They naturally arise as invariants of C^*-algebras. (A C^*-algebra is a norm-closed *-algebra of operators on a Hilbert space - it can be thought of as a noncommutative topological space.) Given a C^*-algebra A, its (concrete) Cuntz semigroup $\mathrm{Cu}(A)$ is constructed from positive elements in matrix algebras over A (and in fact, from the stabilization $A\otimes\mathcal{K}$) in a similar way that the K-theory group $K_0(A)$ is constructed from projections in matrix algebras over A.

The Cuntz semigroup plays an important role in the ongoing program to classify simple, amenable C^*-algebras. This classification program was initiated by George Elliott in the 80s in order to parallel the successful classification of amenable von Neumann algebras. (A von Neumann algebra, or W^*-algebra, is weak*-closed *-algebra of operators on a Hilbert space - it can be thought of as a noncommutative measure space.) The classification of amenable von Neumann algebras, accomplished by Connes, Haagerup and others in the 70s and 80s, is considered one of the greatest accomplishments in operator algebra theory.

The category of Cu-semigroups was introduced in 2008 by Coward, Elliott and Ivancescu, [CEI08]. The morphisms in this category, called Cu-morphisms, are Scott continuous semigroup maps that preserve the way-below relation. From the perspective of domain theory it might seem unusual to insist that morphisms preserve the way-below relation. However, one can show that every *-homomorphism between C^*-algebras, $A \to B$, naturally induces a map between their Cuntz semigroups, $\mathrm{Cu}(A) \to \mathrm{Cu}(B)$, which preserves the way-below relation. Moreover, by considering such morphisms, the category Cu has many desirable properties. For instance, it admits inductive limits (even arbitrary colimits) - a result that is no longer true without requiring that morphisms preserve the way-below relation. Similarly, the category of domains (with Scott continuous maps) does not admit inductive limits, unless one requires the involved maps to preserve the way-below relation.

In [APT14], together with Ramon Antoine and Francesc Perera, we initiated a systematic study of the category Cu. We showed that Cu admits tensor products. The concept of a tensor product is based on the notion of bimorphisms. Given Cu-semigroups S, T and P, a Cu-bimorphism $\varphi\colon S \times T \to P$ is a map that is additive and Scott continuous in each variable, and that preserves the joint way-below relation: If $s' \ll s$ and $t' \ll t$, then $\varphi(s', t') \ll \varphi(s, t)$. The tensor product of Cu-semigroups S and T is a Cu-semigroup $S \otimes T$ together with a universal Cu-bimorphism $S \times T \to S \otimes T$ that linearizes all Cu-bimorphisms from $S \times T$.

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It follows that Cu has the structure of a symmetric, monoidal category: The tensorial unit is $\overline{\mathbb{N}} := \{0, 1, 2, \ldots, \infty\}$, with the obious addition and partial order. Note that $\overline{\mathbb{N}}$ is the Cuntz semigroup of the complex numbers $\mathbb{C}$. Moreover, we have natural isomorphisms

$$(S \otimes T) \otimes P \cong S \otimes (T \otimes P), \quad \text{and} \quad S \otimes T \cong T \otimes S.$$

Given C^*-algebras A and B, there is a natural Cu-morphism

$$\mathrm{Cu}(A) \otimes \mathrm{Cu}(B) \to \mathrm{Cu}(A \otimes B).$$

In some cases (but not always) this map is an isomorphism.

Recently, together with Antoine and Perera, [APT17], we showed that Cu is even a *closed* monoidal category. This means that for Cu-semigroups S and T, there is a Cu-semigroup $[\![S, T]\!]$ playing the role of morphisms from S to T, such that for any other Cu-semigroup P there is a natural bijection

$$\mathrm{Hom}\big(S \otimes T, P\big) \cong \mathrm{Hom}\big(S, [\![T, P]\!]\big).$$

The Cu-semigroup $[\![S, T]\!]$ is a *bivariant* Cu-*semigroup*. These bivariant Cu-semigroups behave very reasonable and provide many new examples of Cu-semigroups. For instance, compact elements in $[\![S, T]\!]$ naturally correspond to Cu-morphisms $S \to T$. (An element a in a Cu-semigroup is called compact if $a \ll a$.)

The step from Cu-semigroups to bivariant Cu-semigroups is similar to the step from K-theory to Kasparov's KK-theory (which is a bivariant version of K-theory). Since KK-theory plays an important role in topology, in index theory, and in the structure theory of C^*-algebras, we expect that bivariant Cuntz semigroups will turn out as a powerful tool in analysis and related areas as well.

References

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