Logics for extended distributive contact lattices

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The notion of contact algebra is one of the main tools in the region based theory of space. It is an extension of Boolean algebra with an additional relation C called contact. The elements of the algebra are called regions and are considered as analogs of physical bodies. Boolean operations are considered as operations for constructing new regions from given ones. The unit element 1 symbolizes the region containing as its parts all regions, and the zero region 0 symbolizes the empty region. There are some problems related to the motivation of the operation of Boolean complementation. A question arises: if the region a in some universe represents a physical body, then what kind of body represents a^*? - it depends on the universe. Because of this in [1] this operation is dropped and the language of distributive lattices is extended by considering as non-definable primitives the relations of contact, nontangential inclusion $\ll$ and dual contact $\widehat{C}$. It has been obtained an axiomatization of the theory consisting of the universal formulas, true in all contact algebras. The structures, satisfying the axioms in question, are called extended distributive contact lattices (EDC-lattices). The well known RCC-8 system of mereotopological relations is definable in the language of EDC-lattices. In [1] are considered also some axiomatic extensions of EDC-lattices yielding representations in T_1 and T_2 topological spaces. In this abstract we consider several logics, corresponding to EDCL. We give completeness theorems with respect to both algebraic and topological semantics for these logics. It turns out that they are decidable.

We consider the quantifier-free first-order language with equality $\mathcal{L}$ which includes:
- constants: 0, 1;
- function symbols: $+$, $\cdot$;
- predicate symbols: $\leq$, C, $\widehat{C}$, $\ll$.

Every EDCL is a structure for $\mathcal{L}$.

We consider the logic L with rule MP and the following axioms:
- the axioms of the classical propositional logic;
- the axiom schemes of distributive lattice;
- the axioms for C, $\widehat{C}$, $\ll$ and the mixed axioms of EDCL - considered as axiom schemes [1].

We consider the following additional rules and an axiom scheme:

(R Ext $\widehat{O}$) $\frac{\alpha\rightarrow(a+p\neq 1\vee b+p=1) \text{ for all variables } p}{\alpha\rightarrow(a\leq b)}$, where α is a formula, a, b are terms

(R U-rich $\ll$) $\frac{\alpha\rightarrow(b+p\neq 1\vee aCp) \text{ for all variables } p}{\alpha\rightarrow(a\overline{\ll}b)}$, where α is a formula, a, b are terms

(R U-rich $\widehat{C}$) $\frac{\alpha\rightarrow(a+p\neq 1\vee b+q\neq 1\vee pCq) \text{ for all variables } p,\ q}{\alpha\rightarrow a\widehat{C}b}$, where α is a formula, a, b are terms

(R Ext C) $\frac{\alpha\rightarrow(p\neq 0\rightarrow aCp) \text{ for all variables } p}{\alpha\rightarrow(a=1)}$, where α is a formula, a is a term

(R Nor1) $\frac{\alpha\rightarrow(p+q\neq 1\vee aCp\vee bCq) \text{ for all variables } p,\ q}{\alpha\rightarrow aCb}$, where α is a formula, a, b are terms

(Con C) $p\neq 0\wedge q\neq 0\wedge p+q=1\rightarrow pCq$

To these rules correspond the additional axioms for EDC-lattices, considered in [1] - the axioms (Ext $\widehat{O}$), (U-rich $\ll$), (U-rich $\widehat{C}$), (Ext C), (Nor1).

Let L' be for example the extension of L with the rule (R Ext $\widehat{O}$) and the axiom scheme (Con C). Then we denote L' by $L_{ConC,Ext\widehat{O}}$ and call the axioms (Con C) and (Ext $\widehat{O}$) corresponding to L' additional axioms. In a similar way we denote any extension of L with some of the considered additional rules and axiom scheme and in a similar way we define its corresponding additional axioms.

Theorem 1 (Completeness theorem with respect to algebraic semantics)**.** *Let L' be some extension of L with zero or more of the considered additional rules and axiom scheme. The following conditions are equivalent for any formula α:*
(i) α is a theorem of L';
(ii) α is true in all EDCL, satisfying the corresponding to L' additional axioms.

We consider the following logics, corresponding to the EDC-lattices, considered in [1]:
1) L;
2) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$;
3) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC}$;
4) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC}$;
5) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},Nor1}$;
6) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC}$;
7) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},Nor1,ConC}$;
8) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,Nor1}$;
9) $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC,Nor1}$.

To every of these logics we juxtapose a class of topological spaces:
1) the class of all T_0, semiregular, compact topological spaces;
2) the class of all T_0, semiregular, compact topological spaces;
3) the class of all T_0, compact, weakly regular topological spaces;
4) the class of all T_0, semiregular, compact, connected topological spaces;
5) the class of all T_0, semiregular, compact, κ - normal topological spaces;
6) the class of all T_0, compact, weakly regular, connected topological spaces;
7) the class of all T_0, semiregular, compact, κ - normal, connected topological spaces;
8) the class of all T_0, compact, weakly regular, κ - normal topological spaces;
9) the class of all T_0, compact, weakly regular, connected, κ - normal topological spaces.

Theorem 2 (Completeness theorem with respect to topological semantics)**.** *Let L' be any of the considered logics. The following conditions are equivalent for any formula α:*
(i) α is a theorem of L';
(ii) α is true in all contact algebras over a topological space from the corresponding to L' class.

Theorem 3. *(i) The logics L, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C}}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},Nor1}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,Nor1}$ have the same theorems and are decidable;*
(ii) The logics $L_{ConC,U-rich\ll}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ConC,Nor1}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC}$, $L_{Ext\widehat{O},U-rich\ll,U-rich\widehat{C},ExtC,ConC,Nor1}$ have the same theorems and are decidable.

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References

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