Weak Fraïssé categories

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We develop category-theoretic framework for the theory of limits of weak Fraïssé classes. Fraïssé theory now belongs to the folklore of model theory, however it actually can easily be formulated in pure category theory, see [3]. The crucial point is the notion of *amalgamation*, saying that two embeddings of a fixed object can be joined by further embeddings into a single one. More precisely, for every two arrows f, g with the same domain there should exist compatible arrows f', g' with the same codomain, such that $f' \circ f = g' \circ g$. A significant relaxing of the amalgamation property, called the *weak amalgamation property* has been discovered by Ivanov [1] and independently by Kechris and Rosendal [2] during their study of generic automorphisms in model theory. It turns out that the weak amalgamation property is sufficient for constructing special objects satisfying certain variant of homogeneity.

Let $\mathfrak{K}$ be a fixed category. We say that $\mathfrak{K}$ has the *weak amalgamation property* (briefly: WAP) if for every $z \in \mathrm{Obj}(\mathfrak{K})$ there exists a $\mathfrak{K}$-arrow $e\colon z \to z'$ such that for every $\mathfrak{K}$-arrows $f\colon z' \to x$, $g\colon z' \to y$ there are $\mathfrak{K}$-arrows $f'\colon x \to w$, $g'\colon y \to w$ satisfying $f' \circ f \circ e = g' \circ g \circ e$. In other words, the square in the diagram

$$\begin{tikzcd}
 & y \arrow[r, "g'"] & w \\
 & z' \arrow[u, "g"] \arrow[r, "f"'] & x \arrow[u, "f'"'] \\
z \arrow[ur, "e"] & &
\end{tikzcd}$$

may not be commutative.

We work within the following setup. Namely, $\mathfrak{K}$ is a fixed category, $\mathfrak{L} \supseteq \mathfrak{K}$ is a bigger category such that $\mathfrak{K}$ is full in $\mathfrak{L}$ and the following conditions are satisfied:

(L0) All $\mathfrak{L}$-arrows are monic.

(L1) Every $\mathfrak{L}$-object is the co-limit of a sequence in $\mathfrak{K}$.

(L2) Every $\mathfrak{K}$-object is ω-small in $\mathfrak{L}$.

We say that $\mathfrak{K}$ is *directed* if for every $x, y \in \mathrm{Obj}(\mathfrak{K})$ there are $v \in \mathrm{Obj}(\mathfrak{K})$ and $\mathfrak{K}$-arrows $i\colon x \to v$, $j\colon y \to v$. In model theory, this is usually called the *joint embedding property*. We say that $\mathfrak{K}$ is *weakly dominated* by a subcategory $\mathfrak{S}$ if the following conditions are satisfied.

(D1) For every $x \in \mathrm{Obj}(\mathfrak{K})$ there is $f \in \mathfrak{K}$ such that $\mathrm{dom}(f) = x$ and $\mathrm{cod}(f) \in \mathrm{Obj}(\mathfrak{S})$.

(D2) For every $y \in \mathrm{Obj}(\mathfrak{S})$ there exists $j\colon y \to y'$ in $\mathfrak{S}$ such that for every $\mathfrak{K}$-arrow $f\colon y' \to z$ there is a $\mathfrak{K}$-arrow $g\colon z \to u$ satisfying $g \circ f \circ j \in \mathfrak{S}$.

We say that $V \in \mathrm{Obj}(\mathfrak{L})$ is $\mathfrak{K}$-*universal* if for every $\mathfrak{K}$-object x there is an $\mathfrak{L}$-arrow from x to V. Finally, we say that $\mathfrak{K}$ is a *weak Fraïssé category* if it is directed, has the WAP, and is weakly dominated by a countable subcategory. An $\mathfrak{L}$-object V is *weakly* $\mathfrak{K}$-*homogeneous* if for every $\mathfrak{K}$-object a there is a $\mathfrak{K}$-arrow $e\colon a \to b$ such that for every $\mathfrak{L}$-arrows $i\colon b \to U$, $j\colon b \to U$ there exists an automorphism $h\colon U \to U$ satisfying $h \circ i \circ e = j \circ e$. This is illustrated in the following diagram in which the triangle is not necessarily commutative.

$$\begin{tikzcd}
 & & U \\
a \arrow[r, "e"] & b \arrow[ur, "j"] \arrow[dr, "i"'] & \\
 & & U \arrow[uu, dashed, "h"]
\end{tikzcd}$$

Below is the main result.

Theorem 1. *Let $\mathfrak{K} \subseteq \mathfrak{L}$ be as above. The following conditions are equivalent.*

(a) *$\mathfrak{K}$ is a weak Fraïssé category.*

(b) *There exists a $\mathfrak{K}$-universal weakly $\mathfrak{K}$-homogeneous object in $\mathfrak{L}$.*

Furthermore, a weakly $\mathfrak{K}$-homogeneous object is unique up to isomorphisms, as long as it exists.

References

[1] A. Ivanov. Generic expansions of ω-categorical structures and semantics of generalized quantifies. *J. Symbolic Logic*, 64:775–789, 1999.

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[3] W. Kubiś. Fraïssé sequences: category-theoretic approach to universal homogeneous structures. *Ann. Pure Appl. Logic*, 165:1755–1811, 2014.