Lawson Topology as the Space of Located Subsets

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This work is a contribution to the constructive theory of Lawson topology, which plays a fundamental role in the theory of topological semilattices [4]. A constructive construction of Lawson topologies (and more generally patch topologies) in the setting of point-free locale theory has already been given by Escardó [3] using the frames of perfect nuclei. There is also a predicative construction by Coquand and Zhang [2] based on entailment relations. However, a geometric notion behind those constructions has not been articulated.

The aim of this work is to clarify the spatial (or geometric) notion behind the Lawson topology on a continuous lattice from a constructive point of view. Our main observation is that the Lawson topology on a continuous lattice has a clear geometric meaning that is of fundamental importance in constructive mathematics, which can be roughly put as follows:

Theorem 1. *The Lawson topology on a continuous lattice is the space of its located subsets.*

In the rest of this abstract, we make the statement of the above theorem precise.

A predicative notion of continuous lattice, *continuous cover*, is given by a triple $(S, \lhd, \mathsf{wb})$ of a set S, a covering relation $\lhd \subseteq S \times \mathcal{P}(S)$, and a base of the way-below relation $\mathsf{wb}\colon S \to \mathcal{P}(S)$ such that

1. $a \in U \implies a \lhd U$;
2. $a \lhd U \;\&\; (\forall b \in U)\, b \lhd V \implies a \lhd V$;
3. $a \lhd \mathsf{wb}(a)$;
4. $a \lhd U \implies (\forall b \in \mathsf{wb}(a))\, (\exists \{a_0, \dots, a_{n-1}\} \subseteq U)\, a \lhd \{a_0, \dots, a_{n-1}\}$.

Then, define a subset $V \subseteq S$ to be *located* if

1. $a \lhd \{a_0, \dots, a_{n-1}\} \;\&\; a \in V \implies (\exists i < n)\, a_i \in V$;
2. $a \in V \implies (\exists b \ll a)\, b \in V$;
3. $a \ll b \implies a \notin V \vee b \in V$,

where $\ll$ is the *way-below* relation:

$$a \ll b \overset{\text{def}}{\iff} (\exists \{a_0, \dots, a_{n-1}\} \subseteq \mathsf{wb}(b))\, a \lhd \{a_0, \dots, a_{n-1}\}.$$

Classically the third condition is superfluous; constructively however, it is non-trivial and of significant importance as the following examples of located subsets suggest; decidable subsets of any set, spreads on the binary tree, Dedekind reals, semi-located subsets of a locally compact metric spaces. See Troelstra and van Dalen [6] and Bishop [1] for details about those examples.

In the setting of continuous cover, one can define an analogue of the notion of Dedekind cut. A pair (L, U) of subsets of S is a *cut* if

1. $a \lhd \{a_0, \dots, a_{n-1}\} \;\&\; a \in U \implies (\exists i < n)\, a_i \in U$;

2. $a \in U \implies (\exists b \ll a)\, b \in U$;
3. $a \lhd \{a_0, \ldots, a_{n-1}\} \subseteq L \implies a \in L$;
4. $a \in L \implies (\exists \{a_0, \ldots, a_{n-1}\} \gg a)\, \{a_0, \ldots, a_{n-1}\} \subseteq L$;
5. $a \ll b \implies a \in L \vee b \in U$;
6. $L \cap U = \emptyset$.

Proposition 2. *There is a bijection between located subsets and cuts.*

The notion of cut is geometric in the sense of propositional geometric theory [7, Chapter 2]. This leads us to the following definition of Lawson topology as a formal space [5].

Definition 3. The *Lawson topology* of a continuous cover $\mathcal{S}$ is the formal space $\mathcal{L}(\mathcal{S})$ presented by the geometric theory $T_{\mathcal{L}}$ whose models are cuts of $\mathcal{S}$.

A *perfect map* between continuous covers $(S, \lhd, \mathsf{wb})$ and $(S', \lhd', \mathsf{wb}')$ is a relation $r \subseteq S \times S'$ such that

1. $a \lhd' U \implies r^- \{a\} \lhd r^- U$;
2. $a \ll' b \implies r^- \{a\} \ll r^- \{b\}$.

Let **CCov** be the category of continuous covers and perfect maps, and let **KReg** be the category of compact regular formal spaces. Then, the universal property of Lawson topology is recovered in the setting of continuous covers.

Theorem 4. *The construction $\mathcal{L}(\mathcal{S})$ extends to a functor $\mathcal{L} \colon$ **CCov** $\to$ **KReg** which is right adjoint to the forgetful functor $U \colon$ **KReg** $\to$ **CCov**.*

The above adjunction induces a monad $K_{\mathcal{L}}$ on **KReg**.

Theorem 5. *The monad $K_{\mathcal{L}}$ induced by the adjunction is naturally isomorphic to the Vietoris monad on* **KReg**.

References

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