States of free product algebras and their integral representation

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In his monograph [6], Hájek established theoretical basis for a wide family of fuzzy (thus, many-valued) logics which, since then, has been significantly developed and further generalized, giving rise to a discipline that has been named as Mathematical Fuzzy logic (MFL). Hájek's approach consists in fixing the real unit interval as standard domain to evaluate atomic formulas, while the evaluation of compound sentences only depends on the chosen operation which provides the semantics for the so called *strong conjunction* connective. His general approach to fuzzy logics is grounded on the observation that, if strong conjunction is interpreted by a continuous t-norm [7], then any other connective of a logic has a natural standard interpretation.

Among continuous t-norms, the so called Łukasiewicz, Gödel and product t-norms play a fundamental role. Indeed, Mostert-Shields' Theorem [7] shows that a t-norm is continuous if and only if it can be built from the previous three ones by the construction of ordinal sum. In other words, a t-norm is continuous if and only if it is an ordinal sum of Łukasiewicz, Gödel and product t-norms. These three operations determine three different algebraizable propositional logics (bringing the same names as their associated t-norms), whose equivalent algebraic semantics are the varieties of MV, Gödel and product algebras respectively.

Within the setting of MFL, *states* were first introduced by Mundici [8] as maps averaging the truth-value in Łukasiewicz logic. In his work, states are functions mapping any MV-algebra $\mathbf{A}$ in the real unit interval $[0,1]$, satisfying a normalization condition and the additivity law. Such functions suitably generalize the classical notion of finitely additive probability measures on Boolean algebras, besides corresponding to convex combinations of valuations in Łukasiewicz propositional logic.

One of the most important results of MV-algebraic state theory is Kroupa-Panti theorem [9, §10], a representation theorem showing that every state of an MV-algebra is the Lebesgue integral with respect to a regular Borel probability measure. Moreover, the correspondence between states and regular Borel probability measures is one-to-one.

Many attempts of defining states in different structures have been made (see for instance [5, §8] for a short survey). In particular, in [2], the authors provide a definition of state for the Lindenbaum algebra of Gödel logic that results in corresponding to the integration of the truth value functions induced by Gödel formulas, with respect to Borel probability measures on the real unit cube $[0,1]^n$. Moreover, such states correspond to convex combinations of finitely many truth-value assignments.

The aim of this contribution is to introduce and study states for the Lindenbaum algebra of product logic, the remaining fundamental many-valued logic for which such a notion is still lacking.

Recall that up to isomorphism (see [1, Theorem 3.2.5]) every element of the free n-generated product algebra $\mathscr{F}_{\mathbb{P}}(n)$ is a *product logic function*, i.e. $[0,1]$-valued function defined on $[0,1]^n$ associated to a product logic formula built over n propositional variables.

Definition 1. A *state of* $\mathscr{F}_{\mathbb{P}}(n)$ will be a map $s : \mathscr{F}_{\mathbb{P}}(n) \to [0,1]$ satisfying the following conditions:

S1. $s(1) = 1$ and $s(0) = 0$,

S2. $s(f \wedge g) + s(f \vee g) = s(f) + s(g)$,

S3. If $f \leq g$, then $s(f) \leq s(g)$,

S4. If $f \neq 0$, then $s(f) = 0$ implies $s(\neg\neg f) = 0$.

By the previous definition, it follows that states of a free product algebra are lattice valuations (axioms S1–S3) as introduced by Birkhoff in [3].

It is worth noticing that product logic functions in $\mathscr{F}_{\mathbb{P}}(n)$ are not continuous, unlike the case of free MV-algebras, and the free n-generated product algebra is not finite, unlike the case of free Gödel algebras. However, there is a *finite* partition of their domain in σ-locally compact subsets, depending on the Boolean skeleton of $\mathscr{F}_{\mathbb{P}}(n)$, upon which the restriction of each product function is continuous. By exploiting this fact, we are able to prove the following integral representation theorem, where we show that our states interestingly represent an axiomatization of the Lebesgue integral as an operator acting on product logic formulas.

Theorem 2 (Integral representation)**.** *For a* $[0,1]$*-valued map* s *on* $\mathscr{F}_{\mathbb{P}}(n)$*, the following are equivalent:*

(i) s *is a state,*

(ii) there is a unique regular Borel probability measure μ *such that, for every* $f \in \mathscr{F}_{\mathbb{P}}(n)$*,*

$$s(f) = \int_{[0,1]^n} f \, \mathrm{d}\mu.$$

Moreover, and quite surprisingly since in the axiomatization of states the product t-norm operation is only indirectly involved via a condition concerning double negation, we prove that every state belongs to the convex closure of product logic valuations. Indeed, in particular, extremal states will result to correspond to the homomorphisms of $\mathscr{F}_{\mathbb{P}}(n)$ into $[0,1]$, that is to say, to the valuations of the logic. Indeed, let $\delta : \mathscr{S}(n) \to \mathscr{M}(n)$ be the map that associates to every state its corresponding measure via Theorem 2.

Theorem 3. *The following are equivalent for a state* $s : \mathscr{F}_{\mathbb{P}}(n) \to [0,1]$

1. s *is extremal;*
2. $\delta(s)$ *is a Dirac measure;*
3. s *is a product homomorphism.*

Thus, since the state space $\mathscr{S}(n)$ is a convex subset of $[0,1]^{\mathscr{F}_{\mathbb{P}}(n)}$, via Krein-Milman Theorem we obtain the following:

Corollary 4. *For every* $n \in \mathbb{N}$*, the state space* $\mathscr{S}(n)$ *is the convex closure of the set of product homomorphisms from* $\mathscr{F}_{\mathbb{P}}(n)$ *into* $[0,1]$.

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References

[1] S. Aguzzoli, S. Bova, B. Gerla. *Free Algebras and Functional Representation for Fuzzy Logics.* Chapter IX of Handbook of Mathematical Fuzzy Logic – Volume 2. P. Cintula, P. Hájek, C. Noguera Eds., Studies in Logic, vol. 38, College Publications, London: 713–791, 2011.

[2] S. Aguzzoli, B. Gerla, V. Marra. *Defuzzifying formulas in Gödel logic through finitely additive measures*. Proceedings of The IEEE International Conference On Fuzzy Systems, FUZZ IEEE 2008, Hong Kong, China: 1886–1893, 2008.

[3] G. Birkhoff. *Lattice Theory*. Amer. Math. Soc. Colloquium Publications, 3rd Ed., Providence, RI, 1967.

[4] T. Flaminio, L. Godo, S. Ugolini. *Towards a probability theory for product logic: states and their integral representation*, submitted.

[5] T. Flaminio, T. Kroupa. *States of MV-algebras*. Handbook of mathematical fuzzy logic, vol 3. (C. Fermüller P. Cintula and C. Noguera, editors), College Publications, London, 2015.

[6] P. Hájek. *Metamathematics of Fuzzy Logics*, Kluwer Academic Publisher, Dordrecht, The Netherlands, 1998.

[7] E. P. Klement, R. Mesiar and E. Pap. *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, 2000.

[8] D. Mundici. *Averaging the truth-value in Łukasiewicz logic*. Studia Logica, 55(1), 113–127, 1995.

[9] D. Mundici. *Advanced Łukasiewicz calculus and MV-algebras*, Trends in Logic 35, Springer, 2011.