Reduced Rickart Rings and Skew Nearlattices

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1 Introduction

Sussman and Subrahmayam proved in [8] and [7] that a certain kind of reduced ring (called *m-domain ring* in [7]) can be decomposed into a collection of disjoint subsets which are closed with respect to multiplication. In [6] it is shown that reduced Rickart rings and m-domain rings are the same thing. This talk is about the order structure of a reduced Rickart ring's decomposition into disjoint semigroups.

Cīrulis proved in [3] that every right normal skew nearlattice can be regarded as a structure called *strong semilattice of semigroups*, and in [5] he shows that any reduced Rickart ring admits a structure of right normal skew nearlattice. It turns out that this strong semilattice of semigroups arises from the semigroup decomposition of [7].

1.1 Reduced Rickart rings

A ring is called *reduced* if it has no nonzero nilpotent elements. It can be easily checked that for all elements x, y of a reduced ring R, $xy = 0$ if and only if $yx = 0$.

The *Abian partial order* on a reduced ring is defined as $x \leq y$ if and only if $xy = xx$. It was proved in [2] that this relation on an arbitrary ring is a partial order if and only if the ring is reduced.

Definition 1.1. [1] A unitary ring R is called a *right Rickart ring* iff for every $a \in R$ there is an idempotent $e \in R$ such that, for all $x \in R$,

$$ax = 0 \quad \text{iff} \quad ex = x.$$

Dually, it is called *left Rickart* iff for every $a \in R$ there is an idempotent $f \in R$ such that, for all $x \in R$, $xa = 0$ iff $xf = x$. A *Rickart ring* is a ring wich is both right and left Rickart.

In a reduced (right or left) Rickart ring R the idempotents e and f from Definition 1.1 are unique and coincide.

1.2 Skew nearlattices

A meet-semilattice is called *nearlattice* if it is finitely bounded complete (i.e., whenever a finite subset has an upper bound, it also has a least upper bound). Skew nearlattices are a generalization of nearlattices. Instead of a meet operation they have an associative and idempotent operation that might not be commutative.

Definition 1.2 ([4, 5])**.** Let S be a finitely bounded complete poset and let $\vee$ denote its join operation. If $*$ is an associative operation on S such that, for all $x, y \in S$, $x \vee y = y$ if and only if $x * y = x$, then the partial algebra $\langle S, *, \vee \rangle$ is called a *(right) skew nearlattice* (see [4]).

For any skew nearlattice $\langle S, *, \vee \rangle$, the reduct $\langle S, * \rangle$ obviously is a band (i.e., an idempotent semigroup). A band $\langle S, * \rangle$ is called *singular* iff $x * y = y$ for all $x, y \in S$ ([3]). A skew nearlattice is called *singular* if the underlying band is singular.

Example 1.3. It was proved in [5] that, given a reduced Rickart ring R equipped with an operation $a \overleftarrow{\wedge} b \coloneqq a''b$, the partial algebra $\left\langle R, \vee, \overleftarrow{\wedge} \right\rangle$ is a skew nearlattice ($\vee$ denotes the join with respect to the natural order of the semigroup $\left\langle R, \overleftarrow{\wedge} \right\rangle$, which coincides with the Abian order). The operation $\overleftarrow{\wedge}$ is therefore called *skew meet*.

Definition 1.4 ([4])**.** Let T be a meet-semilattice and let $\{A_s \mid s \in T\}$ be a family of disjoint semigroups such that, for all $s, t \in T$, the inequality $s \leq t$ implies that there is a semigroup homomorphism $f_s^t : A_t \to A_s$ such that the homomorphisms f_t^t are the identity maps, and for all $r, s, t \in T$, if $r \leq s \leq t$, then $f_s^t f_r^s = f_r^t$.

On the union $A = \bigcup_{s \in T} A_s$ of all the semigroups we define an operation $\overleftarrow{\sqcap}$: If $x \in A_s$ and $y \in A_t$, and $\cdot$ denotes the multiplication of the semigroup $A_{s \wedge t}$, then $x \overleftarrow{\sqcap} y \coloneqq f_{s\wedge t}^s (x) \cdot f_{s \wedge t}^t (y)$. Then we call the algebra $\left\langle A, \overleftarrow{\sqcap} \right\rangle$ a *strong semilattice of the semigroups* $\{A_s\}_{s \in T}$.

2 Skew nearlattices in a reduced Rickart ring

Let U be the set of non-zero-divisors of a reduced Rickart ring R. As shown in [6], we can apply the results on m-domain rings from [7] to R. Therefore we know that the ring R can be decomposed into semigroups of the form Ue (with the usual ring multiplication), where e is an idempotent. Then the set Ue equipped with the skew meet operation $\overleftarrow{\wedge}$, the corresponding partial join operation $\vee$ and the ring multiplication $\cdot$ forms a multiplicative singular skew nearlattice $\left\langle Ue, \vee, \overleftarrow{\wedge}, \cdot \right\rangle$ (i.e., $\left\langle Ue, \vee, \overleftarrow{\wedge} \right\rangle$ is a singular skew nearlattice and $\langle Ue, \cdot \rangle$ is a monoid).

If the semigroups of a strong semilattice of semigroups happen to be multiplicative skew nearlattices and the corresponding semigroup homomorphisms are actually homomorphisms of multiplicative skew nearlattices, then we call this a *strong semilattice of multiplicative skew nearlattices*. Now the whole ring admits such a structure:

Theorem 2.1. *If R is a reduced Rickart ring whose skew meet operation is denoted by $\overleftarrow{\wedge}$, and $\cdot$ is the ring multiplication, then $\left\langle R, \overleftarrow{\wedge}, \cdot \right\rangle$ is a strong semilattice of the multiplicative skew nearlattices $\left\langle Ue, \vee, \overleftarrow{\wedge}, \cdot \right\rangle$.*

There arises the question how much of the structure of a reduced Rickart ring can be "reconstructed" from its strong semilattice of multiplicative skew nearlattices. Given a strong semilattice of multiplicative skew nearlattices that satisfies some additional conditions, we can define a binary operation and constants 0 and 1 on the union of the skew nearlattices such that the resulting algebra is a reduced *Baer semigroup*, i.e., a reduced semigroup with zero such that, for every $a \in S$, there are idempotents $e, f \in S$ such that $ax = 0$ if and only if $ex = x$, and $xa = 0$ if and only if $xf = x$. A Baer semigroup is what is left of a Rickart ring if we "forget" about the addition.

Furthermore, the skew nearlattice of Example 1.3 can be shown to be isomorphic to a skew nearlattice of partial functions.

References

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