Infinitary Propositional Logics and Subdirect Representation

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In this talk we focus on some aspects of the algebraic approach to infinitary propositional logics (for a related development of fuzzy logics and abstract algebraic logic see e.g. [2, 5]). By finitarity, we mean the property of a logic saying that any deduction from a set of premises can be carried out in finitely many steps; infinitary logics are, thus, logics in which some derivations need infinitely many steps (or, equivalently, logics that need some inference rules with infinitely-many premises in their Hilbert-style presentation). Although the majority of logics studied in the literature are finitary, there are prominent natural examples of infinitary ones like the infinitary Łukasiewicz logic of the standard $[0, 1]$ chain or, analogously, the infinitary product logic. For the purpose of this talk, we will refer to them as main examples of infinitary logics.

In [6] we proposed a new hierarchy of infinitary logics based on their completeness properties. Every finitary logic is well-known (see e.g. [5]) to be complete w.r.t. the class of all its relatively subdirectly irreducible models (*RSI-completeness*) and hence also w.r.t. *finitely* relatively subdirectly irreducible models (*RFSI-completeness*). However, not even the later is true for infinitary logics in general. We studied an intermediate (syntactical) notion between finitarity and RFSI-completeness, namely the property that every theory of the logic is the intersection of finitely $\cap$-irreducible theory, i.e. finitely $\cap$-irreducible theories form a basis for all theories. This property is called the *intersection prime extension property* (IPEP) and had already been important in the study of generalized implication and disjunctive connectives (see [1, 4, 3]). The hierarchy also includes a natural stronger extension property that refers to $\cap$-irreducible theories. Their relations are depicted in the figure below. For example both infinitary product and Łukasiewicz logic are shown to have the CIPEP.

$$\begin{tikzcd}
\text{Finitary} \arrow[r, shift left] \arrow[dr, shift left] & \text{IPEP} \arrow[l, shift left, "/" marking] \arrow[r, shift left] \arrow[d, shift left, "/" marking] & \text{RFSI-complete} \arrow[l, shift left, "/" marking] \arrow[d, shift left, "/" marking] \\
 & \text{CIPEP} \arrow[u, shift left] \arrow[ul, shift left, "/" marking] \arrow[r, shift left] & \text{RSI-complete} \arrow[u, shift left] \arrow[l, shift left, "/" marking]
\end{tikzcd}$$

A natural matricial semantics for a propositional logic is that given by its reduced models (which happen to be based on the expected algebras in prominent cases, that is, Boolean algebras for classical logic, Heyting algebras for intuitionistic logic, etc.). For finitary logics such semantics has a powerful property: If L is finitary, then each member of the class of its reduced models $\mathbf{MOD}^*(\mathrm{L})$ can be represented as a subdirect product of reduced subdirectly irreducible models, i.e. $\mathbf{MOD}^*(\mathrm{L}) = \mathbf{P}_{\mathrm{SD}}(\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}})$. This property can be seen as a generalization to matrices of the well-known Birkhoff's representation theorem.

We will discuss the following transferred versions of the syntactical properties: $\mathcal{L}$ has the τ-IPEP whenever for each matrix model $\langle \boldsymbol{A}, F\rangle$ the filter F is the intersection of a collection of (finitely) $\cap$-irreducible L-filters on the algebra $\boldsymbol{A}$, and analogously for τ-CIPEP with $\cap$-irreducible L-filters. Then we can prove the following characterization theorem:

Theorem 1. *For any logic* L *the following are equivalent*

1. L *is protoalgebraic and has the* τ-CIPEP*.*
2. L *is protoalgebraic and the* CIPEP *holds on any free algebra* $\boldsymbol{Fm}_{\mathrm{L}}(\kappa)$*.*
3. *Each member of* $\mathbf{MOD}^*(\mathrm{L})$ *is a subdirect product of subdirectly irreducible members.*

An analogous theorem can be proved for τ-IPEP using finitely subdirectly irreducible models. Also the characterization can written in algebraic terms: κ-generalized quasivarieties are those classes of algebras axiomatized by quasirules with less than κ premises and they are subdirectly representable if and only if an algebraic analog of τ-(C)IPEP holds: The identity congruence on any algebra in the generalized quasivariety can be written as the intersection of a family of $\cap$-irreducible congruences.

We will prove the following about the two mentioned examples of infinitary logics:

1. The infinitary product logic has the CIPEP, but not the τ-IPEP.
2. The infinitary Łukasiewicz logic enjoys the subdirect representation property.

From the first one (which is proved using the fact the each Archimedean product algebras is embeddable into the standard product algebra), we conclude that neither CIPEP nor IPEP imply in general their transferred counterparts. For the proof of the second claim the essential step is to show that any natural extension (variant of the logic with a larger set of variables) is still strongly complete w.r.t. the standard semantics; this involves a topological argument, which is only possible because the connectives of Łukasiewicz logic are continuous w.r.t. the standard interval topology. Observe that the second claim implies that, unlike the other example, the infinitary Łukasiewicz logic has the τ-CIPEP.

Moreover, we will build a variant with rational constants of the infinitary Łukasiewicz logic and show that it has τ-IPEP, i.e. its models are subdirect products of chains, but it is not RSI-complete. Putting all these facts together we will conclude that τ-CIPEP and τ-IPEP are distinct properties and are also different from all the remaining properties seen in the figure, and hence we will obtain a finer hierarchy for infinitary propositional logics.

References

[1] Petr Cintula and Carles Noguera. Implicational (semilinear) logics I: A new hierarchy. *Archive for Mathematical Logic*, 49(4):417–446, 2010.

[2] Petr Cintula and Carles Noguera. A general framework for mathematical fuzzy logic. In Petr Cintula, Petr Hájek, and Carles Noguera, editors, *Handbook of Mathematical Fuzzy Logic - Volume 1*, volume 37 of *Studies in Logic, Mathematical Logic and Foundations*, pages 103–207. College Publications, London, 2011.

[3] Petr Cintula and Carles Noguera. The proof by cases property and its variants in structural consequence relations. *Studia Logica*, 101(4):713–747, 2013.

[4] Petr Cintula and Carles Noguera. Implicational (semilinear) logics II: Additional connectives and characterizations of semilinearity. *Archive for Mathematical Logic*, 55(3):353–372, 2016.

[5] Janusz Czelakowski. *Protoalgebraic Logics*, volume 10 of *Trends in Logic*. Kluwer, Dordrecht, 2001.

[6] Tomáš Lávička and Carles Noguera. A new hierarchy of infinitary logics in abstract algebraic logic. To appear in *Studia Logica*, doi: 10.1007/s11225-016-9699-3.