Intermediate logics admitting structural hypersequent calculi

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A recent trend in proof theory of non-classical logics is to develop systematic and effective procedures to obtain well-behaved proof calculi for uniformly defined classes of non-classical logics. Such procedures will, given a certain kind of specifying data for a logic L, produce an analytic proof calculi with respect to which L is sound and complete. Many procedures fitting this general template already exist, e.g., in the context of sequent calculi for substructural logics [7]; hypersequent calculi for substructural logics [6, 8]; hypersequent calculi for modal logics [11, 12]; labelled sequent calculi for modal and intermediate logics [13, 10] and display calculi for extensions of bi-intuitionistic logic [9].

So far less attention has be given to obtaining negative results demarcating the classes of logics for which such procedures may succeed. See, however, [6, 7, 8] for examples of such negative results. Ideally we would like, given a uniform procedure for obtaining proof calculi of a certain type, a complete classification of the logics for which this procedure may successfully be applied.

We focus on the case of intermediate logics, i.e., consistent extensions of propositional intuitionistic logic **IPC**. For these logics Ciabattoni et al. [6, 8] have isolated a class of axioms, called $\mathcal{P}_3$, which may effectively be translated into so-called structural hypersequent calculi with the property that adding them to the hypersequent calculus **HLJ** for **IPC** preserves cut-admissibility.[1] However, since the class $\mathcal{P}_3$ is not closed under provable equivalence, semantic notions must be introduced in order to determine the class of intermediate logics which can be axiomatised by $\mathcal{P}_3$-formulas and therefore be given cut-free structural hypersequent calculi.

Our contribution consists in introducing criteria for when a given intermediate logic admits a structural hypersequent calculi for which the cut-rule is admissible. These criteria are presented in terms of the algebraic semantics as well as the Kripke semantics. Concretely, we provide the following algebraic characterisation of intermediate logics for which a structural cut-free hypersequent calculus may be provided.

Theorem 1. *An intermediate logic L admits a cut-free structural hypersequent calculus precisely when the corresponding variety of Heyting algebras $\mathbb{V}(L)$ is closed under taking bounded meet-semilattices of its subdirectly irreducible members.*

We note that the requirement that $\mathbf{A} \in \mathbb{V}(L)$, whenever $\mathbf{A}$ is a bounded meet-semilattice of some subdirectly irreducible $\mathbf{B} \in \mathbb{V}(L)$ is a strengthening of the stability condition explored by Bezhanishvili et al. [2, 3, 4, 1]. Our findings may thus be seen as further corroborating the connection between proof-theory and stable logics [5].

[1] In fact, this result by Ciabattoni et al. holds in the more general setting of substructural logics.

Furthermore, we show that any intermediate logic with a structural hypersequent calculus is necessarily sound and complete with respect to an elementary class of Kripke frames. In fact the first-order frame conditions determining such intermediate logics may be classified. These are certain positive Π_2-sentence in the language of Kripke frames, the modal analogue of which are found in the work of Lahav [11] where they are used to construct analytic hypersequent calculi for modal logics.

Finally, our criteria also allow us to show that certain well-known intermediate logics, such as $\mathbf{BD}_n$, for $n \geq 2$, cannot be axiomatised over **HLJ** by structural hypersequent rules.

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