Effect algebras as colimits of finite Boolean algebras *

Gejza Jenča

Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology,Bratislava, Slovak Republic, `gejza.jenca@stuba.sk`

Effect algebras [1] are positive, cancellative, unital partial abelian monoids. The category of effect algebras is denoted by **EA**.

Let us denote the initial segment of natural numbers $\{1, \dots, n\}$ by $[n]$. Note that $[0] = \emptyset$. Let **FinBool** be the full subcategory of the category **FinBool** of Boolean algebras spanned by the set of objects $\{2^{[n]} : n \in \mathbb{N}\}$. **FinBool** is a small, full subcategory of the category of effect algebras.

It was proved by Staton and Uijlen in [6] that every effect algebra A can be faithfully represented by a presheaf $P(A)$ on the category **FinBool**. Explicitely, for an effect algebra A the presheaf $P(A) : \mathbf{FinBool} \to \mathbf{Set}$ maps every object $2^{[n]}$ to the homset $\mathbf{EA}(2^{[n]}, A)$ and every arrow $f : 2^{[n]} \to 2^{[m]}$ the mapping $P(A)(f) : P(2^{[m]}) \to P(2^{[n]})$ defined as the precomposition by f. This determines a functor $P : \mathbf{EA} \to [\mathbf{FinBool}^{op}, \mathbf{Set}]$.

The *category of tests* of an effect algebra A is the category of elements of the presheaf $P(A)$, in symbols $el(P(A))$. We note that every object of $el(P(A))$ is just a morphism of effect algebras $g : 2^{[n]} \to A$ (a *finite observable*) and these are in a one-to-one correspondence with finite sequences $(a_i)_{i\in[n]} \subseteq A$ with $\Sigma_{i\in[n]} a_i = 1$, that are called *tests* [2, 3]. The morphisms then correspond to refinements of tests.

It is clear that for every effect algebra A, there is a functor $D_A : el(P(A)) \to \mathbf{EA}$ that maps every $g : 2^{[n]} \to A$ to its domain $2^{[n]}$. As proved in [6], **FinBool** is a dense subcategory of **EA**. This implies that every effect algebra A is a colimit of its D_A. Moreover, since **EA** is cocomplete [4], we may apply a general argument [5, Theorem I.5.2] to prove that there is a reflection $[\mathbf{FinBool}^{op}, \mathbf{Set}] \to \mathbf{EA}$ left adjoint to P.

Recall, that an effect algebra satisfies the *Riesz decomposition property* (abbreviated by RDP) if and only if, for all u, v_1, v_2 such that $u \le v_1 + v_2$ there are u_1, u_2 such that $u = u_1 + u_2$, $u_1 \le v_1$, $u_2 \le v_2$. Every Boolean algebra and every effect algebra arising from an MV-algebra satisfies the RDP.

Theorem 1. *An effect algebra A satisfies the RDP if and only if every span in $el(P(A))$ can be extended to a commutative square.*

Recall, that an effect algebra is an *orthoalgebra* if and only if, for every element a, the existence of $a + a$ implies that $a = 0$.

Theorem 2. *An effect algebra is an orthoalgebra if and only if for every parallel pair of morphisms in $f_1, f_2 : g \to g'$ in $el(P(A))$ there is a coequalizing morphism $q : g' \to h$ such that $q \circ f_1 = q \circ f_2$.*

Theorem 3. *An effect algebra A is a Boolean algebra if and only if $el(P(A))$ is filtered.*

Let A be an effect algebra. For every Boolean algebra B, a morphism $f : B \to A$ gives rise to a morphism $el(P(f)) : el(P(B)) \to el(P(A))$ in **Cat**. Since $el(P(B))$ is filtered, every such f gives rise to an *ind-object* of the category $el(P(A))$.

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References

[1] Foulis, D., Bennett, M.: Effect algebras and unsharp quantum logics. Found. Phys. **24**, 1325–1346 (1994)

[2] Foulis, D., Randall, C.: Operational quantum statistics. I. Basic concepts. J. Math. Phys. **13**, 1667–1675 (1972)

[3] Gudder, S.: Effect test spaces. Intern. J. Theor. Phys. **36**, 2681–2705 (1997)

[4] Jacobs, B., Mandemaker, J.: Coreflections in algebraic quantum logic. Foundations of physics **42**(7), 932–958 (2012)

[5] MacLane, S., Moerdijk, I.: Sheaves in geometry and logic: A first introduction to topos theory. Springer Science & Business Media (2012)

[6] Staton, S., Uijlen, S.: Effect algebras, presheaves, non-locality and contextuality. In: International Colloquium on Automata, Languages, and Programming, pp. 401–413. Springer (2015)