# The Cuntz semigroup
# and the category Cu

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The Cuntz semigroup of a C*-algebra is an important invariant in the structure and classification theory of C*-algebras. There has been a huge effort towards the classification of these objects over the last 30 years or so, using invariants of K-Theoretical nature. In general, this semigroup captures more information than $K$-theory but is often more delicate to handle. The aim of this talk is to introduce it and discuss various examples as well as its connections with domain theory ([6]).

Very briefly, if $\mathcal{H}$ is a (complex) Hilbert space, let us denote by $\mathbb{B}(\mathcal{H})$ the algebra of bounded, linear operators on $\mathcal{H}$. A C*-algebra $A$ is any norm-closed, involutive subalgebra of $\mathbb{B}(\mathcal{H})$. These objects can also be described abstractly, as follows:

**Definition.** A C*-algebra is a complex Banach algebra $A$, equipped with an involution $^*$ such that $\|a^*a\| = \|a\|^2$ for any $a \in A$. Homomorphisms between C*-algebras are complex algebra maps that respect the involution. (They are automatically continuous.)

The classical definition of the Cuntz semigroup goes back to 1978, and is based on the notion of comparison for positive elements in C*-algebras, as introduced by Cuntz himself ([5]). We review this construction below

**Definition** (The Cuntz semigroup)**.** Let $A$ be a C*-algebra. For positive elements $a, b \in A$, say that $a$ is *Cuntz subequivalent* to $b$ (and write $a \precsim b$) provided there is a sequence $(x_n)$ in $A$ such that $a = \lim_{n\to\infty} x_n b x_n^*$. We say that $a$ and $b$ are *Cuntz equivalent* provided $a \precsim b$ and $b \precsim a$. In symbols, we write $a \sim b$.

Denote by $M_\infty(A) = \cup_{n=1}^\infty M_n(A)$, a directed union of all matrix algebras, and put $W(A) := M_\infty(A)_+/\!\sim$. We denote the class of an element in $W(A)$ by $[a]$, and we define

$$[a] + [b] = [\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)],\ [a] \leq [b] \text{ if and only if } a \precsim b$$

With these operations, $W(A)$ becomes a partially ordered, abelian semigroup, termed the *classical Cuntz semigroup*. The *complete Cuntz semigroup* is constructed in a similar fashion, by replacing $A$ by its tensor product with the compact operators. Namely, it is defined as $\mathrm{Cu}(A) := W(A \otimes \mathbb{K}(\mathcal{H}))$.

In its classical formulation, the Cuntz semigroup may be equipped with an auxiliary relation that makes it into a predomain, in the sense of [7]. This is defined as follows: $[a] \prec [b]$ if and only if $a \precsim (b-\epsilon)_+$ for some $\epsilon > 0$. In the case of the complete Cuntz semigroup, this relation agrees with the sequential way-below relation as in domain theory. In fact, $\mathrm{Cu}(A)$ is an $\omega$-domain, a result that was established by Coward, Elliott, and Ivanescu (see [4]). More concretely, they introduced a category of semigroups, termed Cu and showed the following:

**Theorem.** *For any C*-algebra $A$, the Cuntz semigroup* $\mathrm{Cu}(A)$ *is an object in* Cu*. Moreover, the assignment* $A \mapsto \mathrm{Cu}(A)$ *is a sequentially continuous functor from the category of C*-algebras to the category* Cu.

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The continuity of this functor is very important as many examples in the theory arise as inductive limits. Hence, any valuable invariant must be continuous. That was not the case with the functor $A \mapsto W(A)$. However, this fact can be remedied by looking at the right domain and codomain categories where all the objects belong to. We briefly explain how the category Cu is constructed:

**Definition.** A Cu-*semigroup*, also called *abstract Cuntz semigroup*, is a positively ordered semigroup $S$ that satisfies the following axioms (O1)-(O4):

(O1) Every increasing sequence $(a_n)_n$ in $S$ has a supremum $\sup_n a_n$ in $S$.

(O2) For every element $a \in S$ there exists a sequence $(a_n)_n$ in $S$ with $a_n \ll a_{n+1}$ for all $n \in \mathbb{N}$, and such that $a = \sup_n a_n$.

(O3) If $a' \ll a$ and $b' \ll b$ for $a', b', a, b \in S$, then $a' + b' \ll a + b$.

(O4) If $(a_n)_n$ and $(b_n)_n$ are increasing sequences in $S$, then $\sup_n(a_n + b_n) = \sup_n a_n + \sup_n b_n$.

Morphisms in the category are called Cu-morphisms, that is, maps that preserve addition, order, the zero element, the way-below relation and suprema of increasing sequences. Other maps of interest are the so-called *generalized* Cu-*morphism*, that is, maps as above that do not necessarily preserve the way-below relation.

One of our main results (further developed in [2] and [3]) is the following:

**Theorem** ([1])**.** *The following conditions hold true:*

(i) *There exists a category* W *that admits arbitrary inductive limits and such that the assignment* $A \mapsto W(A)$ *defines a continuous functor from the category* $C^*_{\mathrm{loc}}$ *of local* $C^*$*-algebras to the category* W*.*

(ii) *The category* Cu *is a full, reflective subcategory of* W*. Therefore,* Cu *admits arbitrary inductive limits.*

(iii) *There is a diagram, that commutes up to natural isomorphisms:*

$$\begin{tikzcd}
C^*_{\mathrm{loc}} \arrow[r, "\mathrm{W}"] \arrow[d, bend right, "\gamma"'] & \mathrm{W} \arrow[d, bend left, "\gamma"] \\
C^* \arrow[u, hook] \arrow[r, "\mathrm{Cu}"] & \mathrm{Cu} \arrow[u, hook]
\end{tikzcd}$$

*where* $\gamma\colon \mathrm{W} \to \mathrm{Cu}$ *is the reflection functor and* $\gamma\colon C^*_{\mathrm{loc}} \to C^*$ *is the completion functor that assigns to a local* $C^*$*-algebra its completion (which is a* $C^*$*-algebra). In particular, the assignment* $A \mapsto \mathrm{Cu}(A)$ *is also a continuous functor from the category of* $C^*$*-algebras to the category* Cu *(with respect to arbitrary limits, thus extending the results in [4]).*

(iv) *The category* Cu *is symmetric monoidal.*

# References

[1] Ramon Antoine, Francesc Perera, and Hannes Thiel. Tensor products and regularity properties of Cuntz semigroups. Mem. Amer. Math. Soc. (to appear), preprint (arXiv:1410.0483 [math.OA]), 2014.

[2] Ramon Antoine, Francesc Perera, and Hannes Thiel. Abstract bivariant Cuntz semigroups. preprint (arXiv:1702.01588 [math.OA]), 2017.

[3] Ramon Antoine, Francesc Perera, and Hannes Thiel. Cuntz semigroups of ultraproduct C*-algebras. in preparation, 2017.

[4] Kristofer T. Coward, George A. Elliott, and Cristian Ivanescu. The Cuntz semigroup as an invariant for C*-algebras. *J. Reine Angew. Math.*, 623:161–193, 2008.

[5] Joachim Cuntz. Dimension functions on simple C*-algebras. *Math. Ann.*, 233(2):145–153, 1978.

[6] G. Gierz, K. H. Hofmann, K. Keimel, Jimmie D. Lawson, M. Mislove, and D. S. Scott. *Continuous lattices and domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2003.

[7] Klaus Keimel. The Cuntz semigroup and domain theory. preprint (arXiv:1605.07654 [math.OA]), 2016.