Completenes and Cocompletenes of the category of Cuntz Semigroups

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The category of Cuntz Semigroups, denoted Cu, is a category of ordered commutative monoids with a rich ordered structure. Its introduction and main motivation arises from the Classification program of separable, nuclear C*-algebras, since a certain invariant for an algebra A, namely the *Cuntz semigroup of* A which is denoted Cu(A), has the structure of an object in this category.

After its introduction back in 1978 by J. Cuntz [Cun78], this semigroup received more attention when its non trivial ordered structure was used as a counterexample to existing conjectures regarding classification of C*-algebras (see [Tom08]). Shortly after, and with the aim of using Cu(A) as a classification invariant, Coward, Elliott and Ivanescu [CEI08] proved that Cu(A) had the structure of an ω-domain (with further compatibility properties) and introduced the category Cu as the target for the functor Cu($-$), proving in particular that the category has sequential limits and the functor Cu($-$) is sequentially continuous. This is important since many examples of C*-algebras are built as inductive limits of this kind.

In this note, we improve this result by proving that the category Cu is both complete and cocomplete. As for the functor Cu($-$), not all (co)limits are preserved, but there are some positive results, including a construction of ultraproducts in Cu.

Our approach is to to develop the constructions in some structurally simpler categories, and then use either a reflection or a coreflection functor to define them in Cu. This approach has already been carried out with success in the category Cu for the construction of tensor products (see [APT14]), and resembles the way the tensor product of C*-algebras A, B is carried out: one first develops the *algebraic* tensor product $A\hat{\otimes}B$, then defines there a pseudo-norm, and finally makes the completion.

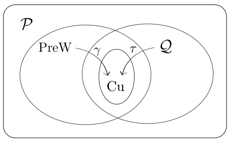
In our case, our simpler categories will be categories of ordered semigroups with an auxiliary relation. In these, the constructions (products, limits, etc..) will extend the set theoretic constructions, and only the appropriate auxiliary relation will have to be chosen. Let us make this concrete:

In [APT14] we introduced a category of ordered semigroups with an auxiliary relation, and satisfying certain axioms, which we termed PreW. The category Cu is then, in a natural way, a full subcategory of PreW when the *way below relation* is chosen as the auxiliary relation. Moreover it was proved ([APT14, Theorem 3.1.10]) that Cu is a reflexive full subcategory of PreW by providing a reflector functor $\gamma\colon$ Prew $\to$ Cu, that is based on the round ideal completion (see [Law97]).

Hence, we obtain a category whose objects are structurally simpler, and from which Cu-semigroups can be obtained though a completion process. It is interesting to note, as observed by K. Keimel in [Kei16], that similar notions had already been around in the field of Domain Theory with different names.

In a similar way, we introduced in [APT17] a different category of ordered semigroups with an auxiliary relation, and again certain properties, which we termed $\mathcal{Q}$. Whereas in PreW we *mainly* relaxed continuity notions, now certain *interpolation* notions are relaxed. Again Cu can naturally be viewed as a full subcategory of $\mathcal{Q}$, and it turns out that a functor $\tau\colon \mathcal{Q} \to \mathrm{Cu}$ can be defined which is now a coreflector, and we have ([APT14, Theorem 3.1.10]) that Cu is a coreflective full subcategory of $\mathcal{Q}$. In this case, we are not aware of a similar or equivalent notion in Domain Theory for the functor τ.

This categories, PreW and $\mathcal{Q}$ can be viewed in a larger category $\mathcal{P}$ (not as full subcategories though), and the constructions γ and τ are naturally equivalent when restricted to the intersection. Moreover, the functor τ, exactly as it is defined, can be extended to $\mathcal{P}$ and then, its restriction to PreW is naturally equivalent to γ. This is clarified in the following diagram.



As mentioned above, doing the necessary constructions in either PreW or $\mathcal{Q}$, and using the fact that Cu is respectively a reflexive or coreflexive full subcategory, we obtain:

The Category of Cuntz semigroups is both complete and cocomplete.

With respect to the question of which of these (co)limit constructions are preserved by the functor, one can not expect a general affirmative answer. There examples of certain pullbacks and certain inverse limits which are not preserved. Nevertheless our techniques allow us to give a positive answer under certain hypothesis. For instance, using the reflector γ above, we prove that $\mathrm{Cu}(-)$ preserves arbitrary inductive limits (see [APT14]).

As a dual example, we can use the coreflector τ to prove that Cu preserves products. Moreover, given an ultrafilter ω in a set I, a notion of ultraproduct can be defined in Cu (as well as for C*-algebras), and prove that, if $(A_i)_{i\in I}$ is a family of C*-algebras then

$$\mathrm{Cu}(\prod_{\omega} A_i) \cong \prod_{\omega} \mathrm{Cu}(A_i).$$

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