Topology as faithful communication through relations

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Most topological concepts can be presented in a predicative and constructive framework, such as that of basic pairs (see [6]). A basic pair $(X, \Vdash, S)$ consists of a set X, a set S and a relation $\Vdash$ from X to S. X represents points, while S is a set of indexes for a basis of neighbourhoods of a topology on X. Elements of S are shortly called observables. An observable a in S is read as an index for the subset $\mathsf{ext}\, a$ of X of those x for which $x \Vdash a$. The presence of S makes the structure of a basic pair *symmetric*. Adding the two axioms

B1) $\mathsf{ext}\, a \cap \mathsf{ext}\, b = \bigcup\{\mathsf{ext}\, c \mid \mathsf{ext}\, c \subseteq \mathsf{ext}\, a \cap \mathsf{ext}\, b\}$

B2) $(\forall x \in X)(\exists a \in S)(x \Vdash a)$

one obtains a predicative and constructive account of topological spaces.

We here add the idea that basic topological concepts, such as closure, interior and continuity, can be characterized as those which can be communicated faithfully between the side of points and the side of observables. This interpretation introduces a new intuitive point of view on topology which can shed light on unexpected links. The foundational framework assumed here is in the common core between the most relevant classical and constructive, predicative and impredicative, foundations, as in [4, 3].

Communication

Suppose an individual A wants to communicate with another individual B, but suppose A and B do not share the same language. However A and B both have their own collection of messages M_A and M_B which they use to represent information. Some messages in M_A are equivalent in the sense that they have the same meaning, and the same for messages in M_B. Hence A is equipped with a pair $(M_A, \sim_A)$ and B with a pair $(M_B, \sim_B)$. If we want A and B to communicate, then

1. B needs a decoding procedure Δ to transform every message in M_A into one of its own messages in M_B. This decoding procedure is good if it translates equivalent messages in M_A into equivalent messages in M_B.

2. Conversely A needs a decoding procedure ∇ to transform every message in M_B into a message in M_A. This decoding procedure is good if it translates equivalent messages in M_B into equivalent messages in M_A.

We can say that a message m in M_A is (faithfully) communicable if it satisfies the following requirement: if A communicates m to B, B translates it obtaining $\Delta(m)$ and then sends $\Delta(m)$ back to A, then the translation $\nabla(\Delta(m))$ by A of $\Delta(m)$ is equivalent to m, that is $\nabla(\Delta(m)) \sim_A m$.

Communication of subsets: interior and closure

Let $(X, \Vdash, S)$ be a basic pair. For all $a \in S$ and $x \in X$, we put $x \in \mathsf{ext}\, a$ if and only $a \in \Diamond x$ if and only if $x \Vdash a$. For all subsets D of X, we put $a \in \Diamond D$ if and only if $\mathsf{ext}\ a \between D$ and $a \in \Box D$ if and only if $\mathsf{ext}\ a \subseteq D$. For all subsets U of S, we put $x \in \mathsf{ext}\, U$ if and only if $\Diamond\, x \between U$ and $x \in \mathsf{rest}\, U$ if and only if $\Diamond\, x \subseteq U$.

Considering X and S as individuals with $(\mathcal{P}(X), =)$ and $(\mathcal{P}(S), =)$ as collections of messages respectively, we prove that for a subset D of X

1. D is open if and only if D is $(\Box, \mathsf{ext})$-communicable;
2. D is closed if and only if D is $(\Diamond, \mathsf{rest})$-communicable.

Communication of relations: continuity

It is natural (see e. g. [1], [2], [5]) to define a *continuous* relation from a basic pair $(X, \Vdash, S)$ to another one $(Y, \Vdash, T)$ as a relation r from X to Y such that for all $b \in T$ and $x \in X$

$$x \in \mathsf{r}^{-}\mathsf{ext}b \to (\exists a \in S)(x \Vdash a \wedge \mathsf{ext}a \subseteq \mathsf{r}^{-}\mathsf{ext}b).$$

One can take (X, Y) and (S, T) as individuals. Their collections of messages are $\mathsf{Rel}(X, Y)$ and $\mathsf{Rel}(S, T)$ equipped with suitable equivalence relations. In this context we prove that there exist natural decoding procedures σ and ρ such that a relation r is continuous if and only if r is (σ, ρ)-communicable.

Communication of elements and functions: convergence

We will finally discuss the notions of convergent subset and convergent relation (see [6]) as notions of communicable element and communicable function respectively.

References

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