Existentially Closed Brouwerian Semilattices

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In algebraic logic some attention has been paid to the class of existentially closed structures in varieties coming from the algebraization of common propositional logics. In fact, there are relevant cases where such classes are elementary: this includes, besides the easy case of Boolean algebras, also Heyting algebras [GZ02], diagonalizable algebras [GZ02] and some universal classes related to temporal logics [GvG16]. This is also true for the variety of Brouwerian semilattices, i.e. the algebraic structures corresponding to the implication-conjunction fragment of intuitionistic logic. Said variety is amalgamable and locally finite, hence by well-known results [Whe76], it has a model completion (whose models are the existentially closed structures). However, very little is known about the related axiomatizations, with the remarkable exception of the case of the locally finite amalgamable varieties of Heyting algebras recently investigated in [DJ10] and the simpler cases of posets and semilattices studied in [AB86]. We use a methodology similar to [DJ10] (relying on classifications of minimal extensions) in order to investigate the case of Brouwerian semilattices. We obtain the finite axiomatization reported below, which is similar in spirit to the axiomatizations from [DJ10] (in the sense that we also have kinds of 'density' and 'splitting' conditions). The main technical problem we must face for this result (making axioms formulation slightly more complex and proofs much more involved) is the lack of joins in the language of Brouwerian semilattices. This investigation also revealed some properties of existentially closed Brouwerian semilattices, namely the nonexistence of meet-irreducible elements, of the minimum and of the joins of incomparable elements, which are suggested and in fact implied by the 'density' and 'splitting' conditions.

Statement of the main result

A *Brouwerian semilattice* is a poset $(P, \leq)$ having a greatest element, inf's of pairs and relative pseudo-complements. We denote the greatest element with 1, the inf of $\{a, b\}$ is called 'meet' of a and b and denoted with $a \wedge b$. The relative pseudo-complement of a and b is denoted with $a \to b$. We recall that $a \to b$ is characterized by the the following property: for every $c \in P$ we have

$$c \leq a \to b \quad \text{iff} \quad c \wedge a \leq b$$

Brouwerian semilattices can also be defined in an alternative way as algebras over the signature $1, \wedge, \to$, subject to the following equations

- $a \wedge a = a$
- $a \wedge b = b \wedge a$
- $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
- $a \wedge 1 = a$
- $a \wedge (a \to b) = a \wedge b$
- $b \wedge (a \to b) = b$

- $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$
- $a \to a = 1$

In case this equational axiomatization is adopted, the partial order $\leq$ is recovered via the definition $a \leq b$ iff $a \wedge b = a$. See [Köh81] for relevant information on Brouwerian semilattices. By a result due to Diego-McKay, Brouwerian semilattices are locally finite (meaning that all finitely generated Brouwerian semilattices are finite); since they are also amalgamable, it follows [Whe76] that the theory of Brouwerian semilattices has a model completion. We prove that such a model completion is given by the above set of axioms for the theory of Brouwerian semilattices together with the three additional axioms (Density1, Density2, Splitting) below.

We use the abbreviation $a \ll b$ for $a \leq b$ and $b \to a = a$.

[Density 1] For every c there exists an element b different from 1 such that $b \ll c$.

[Density 2] For every c, a_1, a_2, d such that $a_1, a_2 \neq 1$, $a_1 \ll c$, $a_2 \ll c$ and $d \to a_1 = a_1$, $d \to a_2 = a_2$ there exists an element b different from 1 such that:

$$\begin{aligned} a_1 &\ll b \\ a_2 &\ll b \\ b &\ll c \\ d \to b &= b \end{aligned}$$

[Splitting] For every a, b_1, b_2 such that $1 \neq a \ll b_1 \wedge b_2$ there exist elements a_1 and a_2 different from 1 such that:

$$\begin{aligned} b_1 \geq a_1 &= a_2 \to a \\ b_2 \geq a_2 &= a_1 \to a \\ a_2 \to b_1 &= b_2 \to b_1 \\ a_1 \to b_2 &= b_1 \to b_2 \end{aligned}$$

Proofs of this and other results can be found in the preliminary manuscript at the following link: http://arxiv.org/abs/1702.08352

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