Types and models in core fuzzy predicate logics

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1 Introduction

The rudiments for the development of the model theory of predicate core fuzzy logics were laid down in [5] with follow up work in places like [3, 2, 4]. A great deal still remains to be done, though. The aim of this talk is to explore the construction of models realizing many and few types in the setting of these logics as well as applications. This kind of problems are well-known from the classical case (cf.[1, 6]).

2 Quick preliminaries

We more or less follow the notation of [3] below. In particular, recall that we write models for our (function symbol-free) predicate language L as structures of the form $(\mathbf{B}, \mathbf{M})$ where $\mathbf{B}$ is an algebra belonging to some variety (which is an extension of the variety of the so called MTL-algebras) corresponding to the logic under consideration and $\mathbf{M}$ is a structure with a domain M and appropriate assignments of truth values to the predicates of the language and of individuals of M to its constants. We write $(\mathbf{B}, \mathbf{M}) \models \phi$ when $||\phi||^{\mathbf{B}}_{\mathbf{M}} = 1$.

Moreover, we are only interested in models where: $||\exists x \phi(x)||^{\mathbf{B}}_{\mathbf{M}} = 1$ means that $||\phi[d]||^{\mathbf{B}}_{\mathbf{M}} = 1$ for some element d of its domain of individuals (call them $\exists$-Henkin models). Henceforth, by a *model* we will always mean one such model.

A *tableau* is going to be a pair (T, U) such that T and U are theories. A tableau is *satisfied* by a model $(\mathbf{B}, \mathbf{M})$, if we have that both $(\mathbf{B}, \mathbf{M}) \models T$ and, for all $\phi \in U$, $(\mathbf{B}, \mathbf{M}) \not\models \phi$. We may define the expression $(T, U) \models \phi$ as meaning that for any model that satisfies (T, U), the model must make ϕ true as well. A tableau (T, U) is said to be *consistent* if $T \vdash \bigvee U_0$ for no finite $U_0 \subseteq U$. In particular, $\bigvee \emptyset$ we define as $\bot$ (semantically, of course, $\bot$ is the l. u. b. of $\emptyset$).

The following result is what we need for our purposes here instead of Theorem 4 from [5].

Theorem 1. (Model Existence Theorem) *Let (T, U) be a consistent tableau. Then there is a model satisfying (T, U).*

Corollary 1. (Tableaux Compactness) *Let (T, U) be a tableau. If every (T_0, U_0), with $|T_0|, |U_0|$ finite and $T_0 \subseteq T$ and $U_0 \subseteq U$, has a model satisfying it, then (T, U) is satisfied in some model.*

3 Models realizing many types

Let $(\mathbf{B}, \mathbf{M})$ be a model. If (p, p') is a tableau in some variable x and parameters in some $A \subseteq M$, we will call p a type of $(\mathbf{B}, \mathbf{M})$ in A if the tableaux $(Th_A(\mathbf{B}, \mathbf{M}) \cup p, \overline{Th}_A(\mathbf{B}, \mathbf{M}) \cup p')$ is satisfiable −where $Th_A(\mathbf{B}, \mathbf{M})$ is the collection of formulas with constants for the elements in A that hold in $(\mathbf{B}, \mathbf{M})$. We will denote the set of all such types by $S^{(\mathbf{B}, \mathbf{M})}(A)$. A model

$(\mathbf{B},\mathbf{M})$ is κ-saturated if for any $A \subseteq M$ suh that $|A| < \kappa$, all $(p,p') \in S^{(\mathbf{B},\mathbf{M})}(A)$ are realized in $(\mathbf{B},\mathbf{M})$.

Theorem 2. *For any* $(\mathbf{B},\mathbf{M})$ *there is a* κ^+*-saturated* L*-elementary extension (in the sense of [5, 3])* $(\mathbf{C},\mathbf{N})$ *of* $(\mathbf{B},\mathbf{M})$*.*

4 Models realizing few types

A pair of sets of formulas (p,p') is a type of a tableau (T,U) if the tableau $(T \cup p, U \cup p')$ is satisfiable.

A type (p,p') of (T,U) is *non-isolated* if for any formulas ϕ, ϕ' such that $(T \cup \{\phi\}, U \cup \{\phi'\})$ is satisfiable, there are $\psi \in p, \psi' \in p'$ such that $(T \cup \{\phi\}, U \cup \{\phi'\}) \nvDash \psi$ or $(T \cup \{\phi, \psi'\}, U \cup \{\phi'\})$ is satisfiable.

Theorem 3. (Omitting types) *Let* (T,U) *be a tableau realized by some model and* (p,p') *a non-isolated* n*-type of* (T,U)*. Then there is model satisfying* (T,U) *which omits* (p,p')*.*

Theorem 4. (Omitting countably many types) *Let* (T,U) *be a tableau realized by some model and* $(p_i,p'_i)(i < \omega)$ *a sequence of non-isolated* n*-types of* (T,U)*. Then there is model satisfying* (T,U) *which omits* $(p_i,p'_i)(i < \omega)$*.*

These omitting types results differ from those in [7] since we are working with tableaux rather than simply theories.

5 Applications

Now we finish with an example of an application of the countable omitting types theorem.

Proposition 1. *Suppose we have binary symbols in our language* $<$ *and* R*. Let* $(\mathbf{B},\mathbf{M})$ *be a countable model of the theory* (Γ,Δ) *where*

$$\Gamma = \{\forall x,y(x<y \vee R(x,y) \vee y<x)\} \cup \{\forall x,y,z(R(x,y) \wedge R(yz) \to R(x,z))\} \cup \{\forall z(\forall x \exists y > x \exists v < z(\psi(v,y)) \to \exists v < z \forall x \exists y > x(\psi(v,y)))\} \cup \{\forall x_0, \ldots x_n \exists y (\textstyle\bigwedge_{i \leq n} x_i < y) : n < \omega\}$$

and

$$\Delta = \emptyset$$

Then there is an L*-elementary extension* $(\mathbf{A},\mathbf{N})$ *of* $(\mathbf{B},\mathbf{M})$*, such that if* $b \in N \setminus M$ *is such that* $R(b,c)$ *does not hold in* $(\mathbf{A},\mathbf{N})$ *for any* $c \in M$*, then, given* $a \in M$*,* $a < b$ *must hold in* $(\mathbf{A},\mathbf{N})$ *(this model might be called an end extension of* $(\mathbf{B},\mathbf{M})$ *relative to* R*).*

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