Expansions of Heyting algebras

Christopher J. Taylor

La Trobe University, Melbourne, Victoria, Australia
`chris.taylor@latrobe.edu.au`

It is well-known that congruences on a Heyting algebra are in one-to-one correspondence with filters of the underlying lattice. If an algebra $\mathbf{A}$ has a Heyting algebra reduct, it is of natural interest to characterise which filters correspond to congruences on $\mathbf{A}$. Such a characterisation was given by Hasimoto [1]. When the filters can be sufficiently described by a single unary term, many useful properties are uncovered. The traditional example arises from boolean algebras with operators. In this setting, an algebra $\mathbf{B} = \langle B; \vee, \wedge, \neg, \{f_i \mid i \in I\}, 0, 1\rangle$ is a *boolean algebra with (dual) operators* (BAO for short) if $\langle B; \vee, \wedge, \neg, 0, 1\rangle$ is a boolean algebra, and for each $i \in I$, the operation f_i is a unary map satisfying $f_i 1 = 1$ and $f_i(x \wedge y) = f_i x \wedge f_i y$. If $\mathbf{B}$ is of finite type, then congruences on $\mathbf{B}$ are determined by filters closed under the map d, defined by

$$dx = \bigwedge\{f_i x \mid i \in I\}.$$

This is easily generalised to the case that each f_i is of any finite arity. The reader is warned that, conventionally, the definition of an operator is dual to the definition given here. However, when the algebra of interest is a Heyting algebra, it turns out that meet-preserving operations are more natural than join-preserving operations. Hasimoto gave a construction which generalises the term above to Heyting algebras equipped with an arbitrary set of arbitrarily many operations (note that Hasimoto uses the word "operator" for an arbitrary unary operation). The construction does not apply in all cases, and even when it does, it does not guarantee that the result is a term function on the algebra. Having said that, natural constraints exist which guarantee both that the construction applies, and produces a term function. As is the case for BAOs, we will restrict our attention to unary operations here and observe that everything is easily generalised to operations of arbitrary arity. In this talk we provide some general conditions which guarantee such a term function. Moreover, provided that the Heyting algebra also includes a dual pseudocomplement operation, we prove that a variety of these algebras is a discriminator variety if and only if it is semisimple, alongside an equational characterisation.

Definition 1.1. We will say that an algebra $\mathbf{A} = \langle A; M, \vee, \wedge, \to, 0, 1\rangle$ is an *expanded Heyting algebra* (EHA for short) if $\langle A; \vee, \wedge, \to, 0, 1\rangle$ is a Heyting algebra, and M is an arbitrary set of unary operations on A. Let $x \leftrightarrow y = (x \to y) \wedge (y \to x)$. We say that a filter $F \subseteq A$ is a *normal filter* (*of* $\mathbf{A}$) provided that, for every $f \in M$, if $x \leftrightarrow y \in F$ then $fx \leftrightarrow fy \in F$. It is easily verified that the set of normal filters of $\mathbf{A}$ forms a complete lattice, and so we will let $\mathbf{Fil}(\mathbf{A})$ denote the lattice of normal filters of $\mathbf{A}$. For all $F \in \mathrm{Fil}(\mathbf{A})$, let $\theta(F)$ be the equivalence relation defined by

$$\theta(F) = \{(x, y) \mid x \leftrightarrow y \in F\}.$$

Theorem 1.2 (Hasimoto [1])**.** *Let $\mathbf{A}$ be an EHA, let F be a normal filter on $\mathbf{A}$, and let α be a congruence on $\mathbf{A}$. Then $\theta(F)$ is a congruence on $\mathbf{A}$. Moreover, the map $\theta\colon \mathbf{Fil}(\mathbf{A}) \to \mathbf{Con}(\mathbf{A})$, defined by $F \mapsto \theta(F)$, is an isomorphism with its inverse given by $\alpha \mapsto 1/\alpha$.*

Definition 1.3. Let $\mathbf{A}$ be an EHA and let t be a unary term in the language of $\mathbf{A}$. We say that t is a *normal filter term* (*on* $\mathbf{A}$) if $t^{\mathbf{A}}$ is order-preserving, and, whenever F is a filter of $\mathbf{A}$, then F is a normal filter of $\mathbf{A}$ if and only if F is closed under $t^{\mathbf{A}}$.

Henceforth we will not be careful to distinguish between terms and term functions. The map d for BAOs seen before is an example of a normal filter term. An easy description of congruences via a normal filter term allows for a deeper investigation of congruence-related properties. In particular, we can characterise equationally definable principal congruences (EDPC) in a very straightforward manner.

Theorem 1.4. *Let $\mathcal{V}$ be a variety of EHAs and assume that t is a normal filter term on $\mathcal{V}$. Let $dx = x \wedge tx$. Then $\mathcal{V}$ has EDPC if and only if there exists $n \in \omega$ such that $\mathcal{V} \models d^{n+1}x = d^n x$.*

Our main result involves dually pseudocomplemented Heyting algebras. A *dual pseudocomplement operation* is an operation $\sim$ such that $x \vee y = 1$ if and only if $y \geq {\sim}x$. If $\mathbf{A}$ is an EHA and there exists $\sim \in M$ such that $\sim$ is a dual pseudocomplement operation, we say that $\mathbf{A}$ is a *dually pseudocomplemented* EHA, and if $M = \{\sim\}$ then $\mathbf{A}$ is a *dually pseudocomplemented Heyting algebra*. Sankappanavar [5] characterised congruences for dually pseudocomplemented Heyting algebras, which is expressed in our terminology by saying that the term $\neg{\sim}$ is a normal filter term (where $\neg x = x \to 0$). Our main result is as follows.

Theorem 1.5 (T., [8])**.** *Let $\mathcal{V}$ be a variety of dually pseudocomplemented EHAs and assume $\mathcal{V}$ has a normal filter term t. Let $dx = x \wedge tx$. Then the following are equivalent:*

1. *$\mathcal{V}$ is semisimple.*
2. *There exists $n \in \omega$ such that $\mathcal{V} \models d^{n+1}x = d^n x$ and $\mathcal{V} \models x \leq d{\sim}d^n\neg x$.*
3. *$\mathcal{V}$ is a discriminator variety.*

Note that the second condition implies EDPC by Theorem 1.4. The argument is based on an argument by Kowalski & Kracht [4] proving the same characterisation for BAOs, which now follows as a corollary of the above theorem. The present author also proved the same characterisation for double-Heyting algebras [7], which also follows. We will also see some new cases for which the characterisation applies to. On the other hand, certain classes of residuated lattices have a suspiciously similar characterisation (Kowalski [2], Kowalski & Ferreirim [3], Takamura [6]), using a similar proof technique, but this theorem does not apply to them. It is believed that this is no coincidence, and further research will attempt to unite these results.

References

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