Definability and conceptual completeness for regular logic

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Regular theories consist of sequents $\varphi \vdash_{\boldsymbol{x}} \psi$, where $\varphi,\ \psi$ are built from atomic formulae by $\wedge$ and $\exists$. Their algebraic counterpart is the notion of regular category, i.e one with finite limits and regular epi - mono factorizations (sufficient for expressing $\exists$) that are stable under pullback ($\exists$ is compatible with substitution of terms). The effectivization $\mathcal{C}_{ef}$ of a regular category $\mathcal{C}$ is the process of universally turning it into an effective (=Barr-exact) one, i.e making every equivalence relation the kernel pair of its coequalizer. It was described in [4] as a full subcategory of the category of sheaves for the subcanonical Grothendieck topology on $\mathcal{C}$ whose coverings are singleton families consisting of regular epis. $\mathcal{C}_{ef}$ has as objects quotients in $\mathrm{Sh}(\mathcal{C}, J)$ of equivalence relations coming from $\mathcal{C}$. Regular functors $F\colon\mathcal{C} \to \mathcal{D}$ between regular categories preserve finite limits and regular epis. Such a functor is covering if for every object $D \in \mathcal{D}$ there is $C \in \mathcal{C}$ and a regular epi $FC \to D$. Regular categories with regular functors are organized in a 2-category REG. $\zeta_{\mathcal{C}}\colon\mathcal{C} \to \mathcal{C}_{ef}$ is the obvious inclusion (restriction of the Yoneda embedding) and we omit it from our notation when it acts on morphisms coming from $\mathcal{C}$. The action of effectivization on a regular functor $F\colon\mathcal{C} \to \mathcal{D}$ is $F^* = F_{ef} :\mathcal{C}_{ef} \to \mathcal{D}_{ef}$ (so that $F^* \cdot \zeta_{\mathcal{C}} \cong \zeta_{\mathcal{D}} \cdot F$). Abusively we may write composites such as $F^*q \cdot Fu$, relying on the latter isomorphism and consistently omitting $\zeta_{\mathcal{C}}$. Our main technical result is the following

Lemma 1. *If $F\colon\mathcal{C} \to \mathcal{D}$ is a full on subobjects regular functor then $F^* = F_{ef} :\mathcal{C}_{ef} \to \mathcal{D}_{ef}$ is also full on subobjects.*

Proof: For a subobject $\sigma\colon S \to F^*X$ the presentation $FC_1 \overset{Fc_0}{\underset{Fc_1}{\rightrightarrows}} FC_0 \xrightarrow{F^*e} F^*X$ of F^*X, arises from the obvious presentation of X in $\mathcal{C}_{ef}$. We pull back the subobject S along F^*e obtaining by our assumption a subobject $Fi\colon FR_0 \to FC_0$, for a subobject $i\colon R_0 \to C_0$, and a regular epimorphism $s\colon FR_0 \to S$.

Let the equivalence relation $(r_0, r_1)\colon R_1 \to R_0 \times R_0$ arise as the intersection of $(c_0, c_1)\colon C_1 \to C_0 \times C_0$ with the subobject $R_0 \times R_0 \to C_0 \times C_0$. Its coequalizer $\zeta_{\mathcal{C}}R_1 \overset{r_0}{\underset{r_1}{\rightrightarrows}} \zeta_{\mathcal{C}}R_0 \xrightarrow{q} Q$ in $\mathcal{C}_{ef}$ gives $S \cong F^*Q$. Indeed we find that $s \cdot Fr_0 = s \cdot Fr_1$, hence a regular epi $r\colon F^*Q \to S$ with $r \cdot F^*q = s$. It is also a mono:

Consider arrows $u_0\ u_1 \colon \zeta_{\mathcal{D}} D \to F^*Q$, such that $r \cdot u_0 = r \cdot u_1$. Since F^*q is a regular epi the generalized elements u_0, u_1 are locally in $\zeta_{\mathcal{D}} D_i$, i.e there is a covering $d' \colon D' \to D$, $i = 0, 1$ and factorizations of $u_i \cdot d' = F^*q \cdot v_i$.

$$\begin{tikzcd}
D' \arrow[r, "d'"] \arrow[dr, shift left, "v_0"] \arrow[dr, shift right, "v_1"'] & D \arrow[r, shift left, "u_0"] \arrow[r, shift right, "u_1"'] & F^*Q \arrow[d, "r"] \\
FR_1 \arrow[r, shift left, "Fr_0"] \arrow[r, shift right, "Fr_1"'] \arrow[d, "Fj"] & FR_0 \arrow[ur, "F^*q"] \arrow[r, "s"] \arrow[d, "Fi"] & S \arrow[d, "\sigma"] \\
FC_1 \arrow[r, shift left, "Fc_0"] \arrow[r, shift right, "Fc_1"'] & FC_0 \arrow[r, "F^*e"] & F^*X
\end{tikzcd}$$

Diagram chasing gives $F^*e \cdot Fi \cdot v_0 = F^*e \cdot Fi \cdot v_1$. The universal properties of (Fc_0, Fc_1) as kernel pair of its coequalizer and of the pullback diagram arising by applying F to the intersection defining (r_0, r_1) give a factorization $\gamma \colon D' \to FC_1$ such that $Fi \cdot v_i = Fc_i \cdot \gamma$, $i = 0, 1$ and, respectively, an $\alpha \colon D' \to FR_1$ such that $(v_0, v_1) = (Fr_0, Fr_1) \cdot \alpha$. Hence $u_0 \cdot d' = u_1 \cdot d'$ and d' is an epi, so $u_0 = u_1$. ∎

Regular functors that are full on subobjects and covering correspond to extensions of theories (of their domain categories) by adding new axioms but no new symbols. Regular functors that are covering, faithful and full on subobjects are full as well. By Lemma 1 and results in [1], such a functor induces a functor with the same properties at the level of effectivizations. By [5], 1.4.9, the induced functor between effectivizations is an equivalence. Hence we have the following strengthening of [6] 2.4.4

Proposition 1. *A regular functor $F \colon \mathcal{C} \to \mathcal{D}$ to an effective category is the effectivization of $\mathcal{C}$ iff it is covering, faithfull and full on subobjects.*

Combining these with results from [2], [3] D3.5.12, we get

Theorem 1. *For $F \colon \mathcal{C} \to \mathcal{D}$ in* REG, *the induced functor between the categories of models $- \cdot F \colon \mathrm{REG}(\mathcal{D}, \mathrm{Set}) \to \mathrm{REG}(\mathcal{C}, \mathrm{Set})$ is fully faithful iff F is full on subobjects and covering.*

References

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