Twist products and dualities

Wesley Fussner

University of Denver, Denver, Colorado, USA
`wesley.fussner@du.edu`

Twist products were originally introduced by Kalman in [6] in the context of lattices enriched with an involution, and have subsequently been employed in different guises by numerous authors (see, e.g., [1, 7, 8] for a sample of the rapidly-growing literature on twist products). Different versions of the twist product construction have often been employed to provide representations of various classes of algebras. In the best cases, twist product constructions participate as one functor witnessing an equivalence between two categories of algebras. On the other hand, such categories sometimes admit topological dualities, such as the Esakia duality for Heyting algebras or Urquhart's duality for algebras associated with relevance logics [9]. Despite its proliferation in algebraic studies, the manner in which the twist product construction manifests on the duals of algebras remains relatively unexplored.

In the present work, we provide a case study illustrating the twist product of dual structures by examining two dualities for the class of bounded Sugihara monoids. These algebras are involutive, idempotent, distributive, bounded commutative residuated lattices, and were shown in [4, 5] to be equivalent to a category of enriched Gödel algebras. One of the functors witnessing this equivalence is a variant of the twist product, and we render this variant as a construction on the topological duals of bounded Sugihara monoids. We call a structure $(X, \leq, X_0, \mathcal{T})$ a *Sugihara space* if

1. $(X, \leq, \mathcal{T})$ is an Esakia space,
2. $(X, \leq)$ is a forest (i.e., for all $x \in X$, the up-set of x is a chain), and
3. $X_0 \subseteq X$ is a clopen collection of $\leq$-minimal elements.

The category of bounded Sugihara monoids is dually equivalent to the category of Sugihara spaces as defined with the appropriate morphisms, and this duality is anchored in the Davey-Werner duality for Kleene algebras [2]. On the other hand, as the equivalent algebraic semantics for the relevance logic **R**-mingle with sentential constant t, the bounded Sugihara monoids also admit a duality in terms of Urquhart's *relevant spaces* [9]. We call the relevant spaces corresponding to bounded Sugihara monoids *Sugihara relevant spaces*, and illustrate a construction that, given a Sugihara space $\mathbf{X}$, produces a Sugihara relevant space $\mathbf{X}^{\bowtie}$. This construction has a much more pictorial character than its analogue on the algebraic side of the duality. Given a Sugihara space $\mathbf{X} = (X, \leq, X_0, \mathcal{T})$, the construction proceeds by producing a copy $-X = \{-x : x \in X \setminus X_0\}$ of those elements outside the designated subset X_0, and defining a new ordering relation $\leq^{\bowtie}$ on $X \cup -X$ by

1. If $x, y \in X$, then $x \leq^{\bowtie} y$ if and only if $x \leq y$,
2. If $-x, -y \in -X$, then $-x \leq^{\bowtie} -y$ if and only if $y \leq x$,
3. If $-x \in -X$ and $y \in X$, then $-x \leq^{\bowtie} y$ if and only if x is $\leq$-comparable to y.

In other words, the underlying poset of $\mathbf{X}^{\bowtie}$ is constructed from $\mathbf{X}$ by doubling the elements $X \setminus X_0$ and reflecting them accross the designated subset X_0.

Having their basis in the Routley-Meyer semantics, Sugihara relevant spaces incorporate a ternary accessibility relation in their signature. This ternary relation realizes the monoid operation of a given bounded Sugihara monoid on its dual space. Capturing the behavior of this relation using only the information encoded in a Sugihara space is a key difficulty in obtaining a dual space analogue of the twist product. It turns out that the appropriate ternary relation R on $X \cup -X$ may be defined in terms of simple conditions on meets and joins of elements of $X \cup -X$, and gives a much simpler presentation of the monoid multiplication than on the algebraic side of the duality. In more detail, define the *absolute value* of an element of $X \cup -X$ by $|x| = x$ if $x \in X$ and $|-x| = x$ if $-x \in -X$. Further, define a partial binary operation $\cdot$ on $X \cup -X$ by

$$x \cdot y = \begin{cases} x \vee y & \text{if } x, y \in X \text{ or } x \parallel y, \text{ provided the join exists} \\ z & \text{if } x \perp y, x \notin X \text{ or } y \notin X, \text{ and } |x| \neq |y|, \text{ where} \\ & z \text{ is whichever of } x, y \text{ has greater absolute value} \\ x \wedge y & \text{if } x \perp y, \text{ and either } x, y \notin X \text{ or } |x| = |y| \\ \text{undefined} & \text{otherwise} \end{cases}$$

where $\perp$ denotes the relation of comparability. The appropriate ternary relation R on $X \cup -X$ is defined by $Rxyz$ if and only if $x \cdot y$ exists and $x \cdot y \leq z$. With this construction, we obtain the following representation theorem.

Theorem 1. *Up to isomorphism, every Sugihara relevant space is of the form* $\mathbf{X}^{\bowtie}$ *for some Sugihara space* $\mathbf{X}$.

Among other things, the above representation theorem explicates the connection between Dunn's Kripke-style semantics for **R**-mingle using a binary accessibility relation [3], and the more usual Routley-Meyer semantics for **R**-mingle using a ternary accessibility relation.

References

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