Topologizing filters on a ring of frictions RS^{-1} and congruence relation on Fil$R$

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This paper introduce the notion of topologizing filters on rings of fractions RS^{-1} for a multiplicative subset S of a commutative ring R. It is shown that the mapping from IdR to IdRS^{-1} given by $I \mapsto IS^{-1}$ induces a map from Fil$R$ to FilRS^{-1}. It is proved that for a multiplicative subset S of a commutative ring R the map $\hat{\varphi}_S : [FilR]^{du} \to [FilRS^{-1}]^{du}$ given by

$$\hat{\varphi}_S(\mathfrak{F}) \stackrel{\text{def}}{=} \{AS^{-1} : A \in \mathfrak{F}\}$$

is an onto homomorphism of lattice ordered monoids. It is proved that for a multiplicative subset S of a commutative ring R, if the monoid operation on Fil$R$ is commutative so is the monoid operation on FilRS^{-1} and if every member of Fil$R$ is idempotent then the same is true of every member of FilRS^{-1}. Moreover, such a map $\hat{\varphi}_S$ gives rise to a canonical congruence relation $\equiv_{\hat{\varphi}_S}$ on Fil$R$ defined by $\mathfrak{F} \equiv_{\hat{\varphi}_S} \mathfrak{G} \Leftrightarrow \hat{\varphi}_S(\mathfrak{F}) = \hat{\varphi}_S(\mathfrak{G})$. The above result tells us that the homomorphism $\hat{\varphi}_S : [FilR]^{du} \to [FilRS^{-1}]^{du}$ restricts to a homomorphism from the Jansian topologizing filters of Fil$R$ onto the Jansian topologizing filters of FilRS^{-1}.

It is proved that for a commutative ring R for which Fil$R$ is commutative, then $\bigcap\{\equiv_{\hat{\varphi}_{S_P}} : P \in \operatorname{Spec_m} R\}$ is the identity congruence on Fil$R$, that is, for all $\mathfrak{F}, \mathfrak{G} \in FilR$,

$$\mathfrak{F} = \mathfrak{G} \Leftrightarrow \mathfrak{F} \equiv_{\hat{\varphi}_{S_P}} \mathfrak{G} \ \ \forall P \in \operatorname{Spec_m} R.$$

As one of the main results of this paper it is shown that if R is a commutative ring for which Fil$R$ is commutative, then the previous result yields the following subdirect decomposition:

$$[FilR]^{du} \cong [FilR]^{du} / \Big(\bigcap\nolimits_{P \in Spec_m R} \equiv_{\hat{\varphi}_{S_P}}\Big) \hookrightarrow \prod_{P \in Spec_m R} ([FilR]^{du} / \equiv_{\hat{\varphi}_{S_P}}) \cong \prod_{P \in Spec_m R} [FilR_P]^{du}.$$

For an arbitrary ring R for which $[\operatorname{Fil} R_R]^{\mathrm{du}}$ is two-sided residuated, it is shown that R satisfies the DCC on left annihilator ideals, and the ACC on right annihilator ideals. It is well-known that a commutative noetherian ring has finitely many minimal prime ideals and as an extension of this, it is proved that if R is an arbitrary ring for which $[Fil\, R]^{du}$ is two-sided residuated, then R contains finitely many minimal prime ideals. It is also shown that for a Prüfer domain R for which $\operatorname{Fil} R$ is commutative, R_P is a (noetherian) rank 1 discrete valuation domain for every maximal ideal P of R.

This paper is concluded by proving that for a Prüfer domain R, Fil$R$ is commutative if and only if R is noetherian and thus a Dedekind domain which extends a known result which says that a valuation domian for which Fil$R$ is commutative is noetherian and thus rank 1 discrete.

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