Analogies between small scale topology and large scale topology from the nonstandard perspective

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Large scale topology is the study of the large scale (asymptotic) behaviour of various spaces. It is well-known that there are many analogies between small scale topology and large scale topology. Our contribution is to study these analogies in the light of nonstandard analysis.

Let $\mathbb{U}$ be a transitive universe that satisfies sufficiently many axioms of ZFC and has all standard objects we need. We fix an enlargement $*\colon \mathbb{U} \hookrightarrow {}^{*}\mathbb{U}$ of $\mathbb{U}$. A formula is said to be Π_1^{st} if it is of the form $\forall x \in \mathbb{U}.\varphi(x, \vec{a})$, where φ is a $\in$-formula and $\vec{a}$ is parameters from $\mathbb{U}$. A formula is said to be Σ_1^{st} if it is of the form $\exists x \in \mathbb{U}.\varphi(x, \vec{a})$, where φ and $\vec{a}$ are the same as above.

Let X be a topological space with a topology $\mathcal{O}_X$. The *monad* of $x \in X$ is the Π_1^{st}-set $\mu_X(x) := \bigcap_{x \in U \in \mathcal{O}_X} {}^{*}U$. The monad map $\mu_X\colon X \to \mathcal{P}({}^{*}X)$ uniquely determines the topology $\mathcal{O}_X$. Next, let X be a uniform space with a uniformity $\mathcal{U}_X$. The *infinite closeness relation* on ${}^{*}X$ is the Π_1^{st}-equivalence relation defined by $\approx_X := \bigcap_{E \in \mathcal{U}_X} {}^{*}E$. Like topological spaces, the infinite closeness relation $\approx_X$ uniquely determines the uniformity $\mathcal{U}_X$ ([1]). Thus we can consider small scale topology as the study of Π_1^{st}-sets.

Let X be a bornological space with a bornology $\mathcal{B}_X$. In our setting, a bornology on X is defined to be a nonempty cover of X that is closed under taking subsets and finite *nondisjoint* unions. Bornology is a minimal framework in which we can discuss boundedness. For more details, see Hogbe-Nlend [2]. The *galaxy* of $x \in X$ is defined as the Σ_1^{st}-set $G_X(x) := \bigcup_{x \in B \in \mathcal{B}_X} {}^{*}B$. We show that the galaxy map $G_X\colon X \to \mathcal{P}({}^{*}X)$ uniquely determines the bornology $\mathcal{B}_X$. Next, let X be a coarse space with a coarse structure $\mathcal{E}_X$. The *finite closeness relation* on ${}^{*}X$ is defined as the Σ_1^{st}-equivalence relation $\sim_X := \bigcup_{E \in \mathcal{E}_X} {}^{*}E$. We show that the finite closeness relation $\sim_X$ uniquely determines the coarse structure $\mathcal{E}_X$. Similarly to small scale, we can think of large scale topology as the study of Σ_1^{st}-sets. In this sense, large scale topology is the logical dual of small scale topology.

Many small scale concepts topology have nonstandard characterisations in terms of monad and infinite closeness (see Robinson [3] and Stroyan and Luxemburg [4]). For example,

- a map $f\colon X \to Y$ between topological spaces is continuous at $x \in X$ if and only if ${}^{*}f(\mu_X(x)) \subseteq \mu_Y(f(x))$;
- a map $f\colon X \to Y$ between uniform spaces is uniformly continuous if and only if for every $x, y \in {}^{*}X$, if $x \approx_X y$, then ${}^{*}f(x) \approx_Y {}^{*}f(y)$;
- a family $\mathcal{F}$ of maps between uniform spaces X, Y is uniformly equicontinuous if and only if for any $f \in {}^{*}\mathcal{F}$ and $x, y \in {}^{*}X$, if $x \approx_X y$, then $f(x) \approx_Y f(y)$.

As the large scale analogues, we obtain the following nonstandard characterisations of large scale concepts in terms of galaxy and finite closeness:

- a map $f\colon X \to Y$ between bornological spaces is bornological at $x \in X$ if and only if ${}^{*}f(G_X(x)) \subseteq G_Y(f(x))$;

- a map $f\colon X \to Y$ between coarse spaces is bornologous if and only if for every $x, y \in {}^{*}X$, if $x \sim_X y$, then ${}^{*}f(x) \sim_Y {}^{*}f(y)$;
- a family $\mathcal{F}$ of maps between coarse spaces X, Y is uniformly equibounded if and only if for any $f \in {}^{*}\mathcal{F}$ and $x, y \in {}^{*}X$, if $x \sim_X y$, then $f(x) \sim_Y f(y)$.

References

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