Large scale topology is the study of the large scale (asymptotic) behaviour of various spaces. It is well-known that there are many analogies between small scale topology and large scale topology. Our contribution is to study these analogies in the light of nonstandard analysis.

Let $U$ be a transitive universe that satisfies sufficiently many axioms of ZFC and has all standard objects we need. We fix an enlargement $*: U \rightarrow \ast U$ of $U$. A formula is said to be $\Pi^1_n$ if it is of the form $\forall x \in U, \varphi(x, \bar{a})$, where $\varphi$ is a $\varepsilon$-formula and $\bar{a}$ is parameters from $U$. A formula is said to be $\Sigma^1_n$ if it is of the form $\exists x \in U, \varphi(x, \bar{a})$, where $\varphi$ and $\bar{a}$ are the same as above.

Let $X$ be a topological space with a topology $\mathcal{O}_X$. The monad of $x \in X$ is the $\Pi^1_1$-set $\mu_X(x) := \bigcap_{U \in \mathcal{O}_X} \ast U$. The monad map $\mu_X : X \rightarrow \mathcal{P}(\ast X)$ uniquely determines the topology $\mathcal{O}_X$. Next, let $X$ be a uniform space with a uniformity $\mathcal{U}_X$. The infinite closeness relation on $\ast X$ is the $\Pi^1_1$-equivalence relation defined by $\approx_X := \bigcap_{E \in \mathcal{U}_X} \ast E$. Like topological spaces, the infinite closeness relation $\approx_X$ uniquely determines the uniformity $\mathcal{U}_X$ ([1]). Thus we can consider small scale topology as the study of $\Pi^1_1$-sets.

Let $X$ be a bornological space with a bornology $\mathcal{B}_X$. In our setting, a bornology on $X$ is defined to be a nonempty cover of $X$ that is closed under taking subsets and finite nondisjoint unions. Bornology is a minimal framework in which we can discuss boundedness. For more details, see Hogbe-Nlend [2]. The galaxy of $x \in X$ is defined as the $\Sigma^1_1$-set $G_X(x) := \bigcup_{E \in \mathcal{B}_X} \ast E$. We show that the galaxy map $G_X : X \rightarrow \mathcal{P}(\ast X)$ uniquely determines the bornology $\mathcal{B}_X$. Next, let $X$ be a coarse space with a coarse structure $\mathcal{E}_X$. The finite closeness relation on $\ast X$ is defined as the $\Sigma^1_1$-equivalence relation $\sim_X := \bigcup_{E \in \mathcal{E}_X} \ast E$. We show that the finite closeness relation $\sim_X$ uniquely determines the coarse structure $\mathcal{E}_X$. Similarly to small scale, we can think of large scale topology as the study of $\Sigma^1_1$-sets. In this sense, large scale topology is the logical dual of small scale topology.

Many small scale concepts topology have nonstandard characterisations in terms of monad and infinite closeness (see Robinson [3] and Stroyan and Luxemburg [4]). For example,

- a map $f : X \rightarrow Y$ between topological spaces is continuous at $x \in X$ if and only if $\ast f(\mu_X(x)) \subseteq \mu_Y(f(x))$;
- a map $f : X \rightarrow Y$ between uniform spaces is uniformly continuous if and only if for every $x, y \in \ast X$, if $x \approx_X y$, then $\ast f(x) \approx_Y \ast f(y)$;
- a family $\mathcal{F}$ of maps between uniform spaces $X, Y$ is uniformly equicontinuous if and only if for any $f \in \ast \mathcal{F}$ and $x, y \in \ast X$, if $x \approx_X y$, then $f(x) \approx_Y f(y)$.

As the large scale analogues, we obtain the following nonstandard characterisations of large scale concepts in terms of galaxy and finite closeness:

- a map $f : X \rightarrow Y$ between bornological spaces is bornological at $x \in X$ if and only if $\ast f(G_X(x)) \subseteq G_Y(f(x))$;
• a map $f : X \to Y$ between coarse spaces is bornologous if and only if for every $x, y \in ^*X$, if $x \sim_X y$, then $^*f(x) \sim_Y ^*f(y)$;

• a family $\mathcal{F}$ of maps between coarse spaces $X, Y$ is uniformly equibounded if and only if for any $f \in ^*\mathcal{F}$ and $x, y \in ^*X$, if $x \sim_X y$, then $f(x) \sim_Y f(y)$.

References


