Multiplicative derivations of commutative residuated lattices

Michiro Kondo*

Tokyo Denki University, Tokyo, Japan
mkondo@mail.dendai.ac.jp

1 Introduction

For a theory of algebras with two operations $+$ and $\cdot$, we have an interesting method, *derivations*, to develop the structure theory, as an analogy of derivations of analysis. The notion of derivation of algebras was firstly applied to the theory of ring ([4]), and after that it was also applied to other algebras, such as lattices ([2, 6]) and MV-algebras ([1, 7]). We here aplied the derivation theory to the (commutative) residuated lattices which are very basic algebtras corresponding to fuzzy logic. For a residuated lattice L, a map $d : L \to L$ is called a derivation in [3] if it satisfies the condition: For all $x, y \in L$,

$$d(x \odot y) = (dx \odot y) \vee (x \odot dy).$$

Let L be a commutative residuated lattice and d be a good ideal derivation and F a d-filter of L, which are defined later. We show that

(1) The set $\mathrm{Fix_d}(L)$ of all fixed points of d forms a residuated lattice and $L/\ker d \cong \mathrm{Fix_d}(L)$.

(2) A map $d/F : L/F \to L/F$ defined by $(d/F)(x/F) = dx/F$ is also a good ideal derivation of L/F.

(3) The quotient residuated lattices $\mathrm{Fix_{d/F}}(L/F)$ and $\mathrm{Fix_d}(L)/d(F)$ are isomorphic, namely,

$$\mathrm{Fix_{d/F}}(L/F) \cong \mathrm{Fix_d}(L)/d(F).$$

2 Derivations of residuated lattices

Let $L = (L, \wedge, vee, \to 0, 1)$ be a (commutative) residuated lattice and $B(L)$ be the set of all complemented elements of L. We define derivations of residuated lattices according to [3]. A map $d : L \to L$ is called a *multiplicative derivation* (or simply *derivation*) of L if it satisfies the condition

$$d(x \wedge y) = (dx \odot y) \vee (x \odot dy) \quad (\forall x, y \in L)$$

A derivation d is called *ideal* if $x \le y$ then $dx \le dy$ and $dx \le x$ for all $x, y \in L$. Moreover, a derivation d is said to be *good* if $d1 \in B(L)$.

Theorem 1 ([3]). *Let d be a derivation of L and $d1 \in B(L)$. Then the following are equivalent: for all $x, y \in L$,*

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(1) *d is an ideal derivation;*

(2) $dx \leq d1$*;*

(3) $dx = x \odot d1$*;*

(4) $d(x \wedge y) = dx \wedge dy$*;*

(5) $d(x \vee y) = dx \vee dy$*;*

(6) $d(x \odot y) = dx \odot dy$*.*

For a derivation d of L, we consider a subset $\mathrm{Fix_d}(L) = \{x \in L \mid dx = x\}$ of the set of all fixed elements of L for d.

Proposition 1. *For a good ideal derivation d, we have* $\mathrm{Fix_d}(L) = d(L)$.

We have

Theorem 2. $\mathrm{Fix_d}(L) = (\mathrm{Fix_d}(L), \wedge, \vee, \odot, \mapsto, 0, d1)$ *is a residuated lattice, where operations on* $\mathrm{Fix_d}(L)$ *are defined as follows:*

$$dx \wedge dy = d(x \wedge y) \qquad dx \vee dy = d(x \vee y)$$
$$dx \odot dy = d(x \odot y) \qquad dx \mapsto dy = d(dx \to dy).$$

A filter F is called a *d-filter* if $x \in F$ implies $dx \in F$ for all $x \in L$. It is easy to show that a quotient structure L/F is also a residuated lattice for a filter F. Moreover, we have the following.

Proposition 2. *Let d be a good ideal derivation and F be a d-filter of L. A map $d/F : L/F \to L/F$ defined by $(d/F)(x/F) = dx/F$ for all $x/F \in L/F$ is a good ideal derivation of L/F.*

Therefore, the quotient structure $(d/F)(L/F) = \mathrm{Fix_{d/F}}(L/F)$ is a residuated lattice. Since F is a d-filter of L, $d(F)$ is also a filter of $d(L)$ and thus $d(L)/d(F)$ forms a residuated lattice. It is natural to ask what the relation between two residuated lattices $(d/F)(L/F)$ and $d(L)/d(F)$ is. Next result is an answer.

Theorem 3. *Let d be a good ideal derivation and F be a d-filter of L. Then we have* $(d/F)(L/F) = \mathrm{Fix_{d/F}}(L/F)$ *is isomorphic to* $d(L)/d(F)$*, that is,*

$$(d/F)(L/F) = \mathrm{Fix_{d/F}}(L/F) \cong d(L)/d(F).$$

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