Bicategory of Theories as an Approach to Model Theory

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First-order categorical logic (FOCL for short) originated as a categorical foundation for model theory (in Makkai & Reyes [6]). Some classical model-theoretic phenomena can be efficiently described in terms of FOCL. However, most concepts in modern model theory remain to be under categorical consideration. Our present aim is to set up a framework suitable for comprehensive categorical analysis of model theory. In this talk, we work on the notion of "category of theories," whose importance will be discussed below.

From the viewpoint of FOCL, classical first-order theories give rise to *Boolean pretoposes*, i.e. categories equipped with logical operations and quotients of equivalence relations. They are called *classifying pretoposes* of theories. As Harnik [5] pointed out, a construction of classifying pretoposes can be given via Shelah's eq-construction. Moreover, any Boolean pretopos arises (up to categorical equivalence) as a classifying pretopos of some classical theory.

Since mathematical objects often constitute a category, it is natural to ask what morphisms between theories are. It has been observed that *interpretations* between theories (in the model-theoretic sense) induce pretopos functors, i.e. functors preserving pretopos structures, between corresponding classifying pretoposes. So, among categorical logicians, there exists a common sense that

the (2-)category $\mathfrak{BPretop}_*$ of Boolean pretoposes, pretopos functors and natural isomorphisms can be regarded as a "category of theories,"

while no purely syntactic definition of "category of theories" is widely accepted.

Our approach is as follows: once we have defined *homotopies* between interpretations, we obtain a *bicategory* $\mathfrak{Th}$ which consists of theories, interpretations and homotopies. We show that the construction of classifying pretoposes gives a pseudofunctor $\mathfrak{Th} \to \mathfrak{BPretop}_*$. In fact, it is a biequivalence, and hence our definition of $\mathfrak{Th}$ is consistent with the above consensus.

We also make a close observation on (internal) equivalences in $\mathfrak{Th}$. In the model-theoretic context, these equivalences are called *bi-interpretations*. Via the biequivalence above, existence of a bi-interpretation between two theories coincides with *Morita equivalence*, i.e. categorical equivalence between corresponding classifying pretoposes. We give another characterization of bi-interpretability (and Morita equivalence) by using the notion of *Morita extension*, recently introduced by Barrett & Halvorson [2], which is a slight generalization of definitional extension admitting sort definitions. We also give a simple proof for Tsementzis' syntactic characterization of Morita equivalence [7].

Future directions. We believe that this framework will promote more extensive uses of categories in modern model theory. We indicate the following lines of research:

- Using preceding works on dualities in first-order logic (e.g. Caramello's [3] and Forssell's [4, 1]), we will consider relationships between various mathematical objects associated with theories. Examples of such mathematical objects include classifying (pre)toposes, categories of models and topological groupoids of models and isomorphisms.

- Certain model-theoretic constructions of theories, e.g. elementary diagrams and nonforking extensions of types, can be interpreted as categorical constructions in the bicategory

$\mathfrak{Th}$ of theories or other related 2-categories. Moreover, we also expect that category theory will give new constructions of theories.

References

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