Q-sup-algebras and their representation

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The topic of sets with fuzzy order relations valuated in complete lattices with additional structure has been quite active in the recent decade, and a number of papers have been published (see [3, 5, 6] among many others).

Based on a quantale-valued order relation and subset membership, counterparts to common order-theoretic notions can be defined, like monotone mappings, adjunctions, joins and meets, complete lattices, or join-preserving mappings, and one can consider a category formed from the latter two concepts. An attempt for systematic study of such categories of fuzzy complete lattices with quantale valuation ("Q-sup-lattices") with fuzzy join-preserving mappings has been made by the second author in his recent paper [8].

With some theory of Q-sup-lattices available, new concepts of algebraic structures in this category can easily be built. In this paper, we shall deal with general algebras with finitary operations, building on existing results obtained for algebras based on crisp sup-lattices ('sup algebras' as in [1, 7]). We can see [9] that our fuzzy structures behave in strong analogy to their crisp counterparts.

We also highlight an important fact: that concepts based on a fuzzy order relation (in the sense of the quantale valuation as studied in this work) *should not be treated as generalizations of their crisp variants* – they are rather standard crisp concepts of order theory, satisfying certain additional properties. This fact also reduces the work needed to carry out proofs. Thus, even with the additional properties imposed, the theory of fuzzy-ordered structures develops consistently with its crisp counterpart.

The connection between fuzzy and crisp order concepts has also been justified by I. Stubbe in a general categorial setting of modules over quantaloids [4], and in the recent work of S. A. Solovyov in the quantale-fuzzy setting [3] where categories of quantale-valued sup-lattices are proved to be isomorphic to well-investigated categories of quantale modules. This isomorphism will enable us to make direct transfer of some of the fundamental constructions and properties known for quantale modules, to our framework. The bridge between these two worlds allows us to open a space for surprising interpretations.

With this paper we hope to contribute to the theory of quantales and quantale-like structures. It considers the notion of Q-sup-algebra and shows a representation theorem for such structures generalizing the well-known representation theorems for quantales, sup-algebras and quantale algebras [2].

Theorem 1. *If $(A, \bigsqcup_A, \Omega)$ is a Q-sup-algebra, then*

1. *Q^A can be equipped with a structure of a Q-sup-algebra.*
2. *There is a nucleus j on Q^A such that $A \cong Q^A_j$.*

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In addition, we present some important properties of the category of Q-sup-algebras.

Theorem 2. *The category of Q-sup-algebras is a monadic construct.*

Corollary 3. *The category of Q-sup-algebras is complete, cocomplete, wellpowered, extremally co-wellpowered, and has regular factorizations. Moreover, monomorphisms are precisely those morphisms that are injective functions.*

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