On Two Approaches to Concrete Dualities and Their Relationships

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There are two categorical approaches to the unification of the dualities between various kinds of algebraic structures and of topological structures. The present study investigates how these essentially different approaches are related, and applies the presented results to the categories of certain types of universal algebras (including infinitary operations).

1 Introduction

Duality of algebraic structures (e.g., Boolean algebras, distributive lattices and Heyting algebras) with topological structures (e.g., Stone spaces, Priestly spaces and Heyting spaces) has been a major issue in topology, algebra and logic [1, 2, 6, 8]. There are two general and categorical approaches [2, 8] to the duality issue: The first approach operates with concrete categories **C** and **D** over the category **Set** of sets and functions. Schizophrenic object is the key concept of this approach [8] defined as a triple $(\tilde{C}, s, \tilde{D})$ provided that \tilde{C} is a **C**-object, \tilde{D}

is a **D**-object, s is a bijective function from the underlying set of \hat{C} to the underlying set of \hat{D} , and two additional conditions are satisfied. Such a schizophrenic object determines an adjoint situation

$$(\gamma, \alpha): S \dashv T: \mathbf{C}^{op} \to \mathbf{D}.$$
(1)

If we consider the full subcategory $Fix(\alpha)$ of **C** with those **C**-objects A for which the Ath component α_A of α is an isomorphism in \mathbf{C}^{op} , and similarly, the full subcategory $Fix(\gamma)$ of **D** with respect to γ , then the adjoint situation (1) restricts to a duality between $Fix(\alpha)$ and $Fix(\gamma)$, which is the main result of the first approach describing many existing dualities, e.g., Stone, Priestley and localic dualities.

In the second approach [2], **C** is taken as an abstract category, which is not necessarily a concrete category over **Set**. As a formulation of fixed-basis fuzzy topological spaces in the category **C** with set-indexed products, **C**- \mathcal{M} -L-spaces are defined in this approach to be pairs $\left(X, \tau \xrightarrow{m} L^X\right)$ consisting of a set X and an \mathcal{M} -morphism $\tau \xrightarrow{m} L^X \in \mathcal{M}$, where L is an arbitrarily fixed object of **C**, L^X is an Xth power of L and \mathcal{M} is a class of **C**-monomorphisms.

C- \mathcal{M} -L-spaces and C- \mathcal{M} -L-continuous functions form a category C- \mathcal{M} -L-Top, which, under the assumption of C being essentially $(\mathcal{E}, \mathcal{M})$ -structured, relates to C with the adjoint situation

$(\eta, \varepsilon) : L\Omega_{\mathcal{M}} \dashv LPt_{\mathcal{M}} : \mathbf{C}^{op} \to \mathbf{C}\text{-}\mathcal{M}\text{-}L\text{-}\mathbf{Top}.$

This adjoint situation gives rise to a duality between the full subcategory $\mathbf{SPA}(\mathbf{C})$ of \mathbf{C} with *L*-spatial objects and the full subcategory $\mathbf{SOBTop}(\mathbf{C})$ of \mathbf{C} - \mathcal{M} -*L*- \mathbf{Top} with *L*-sober objects, where *L*-spatiality of a \mathbf{C} -object *A* means $\varepsilon_A \in Iso(\mathbf{C}^{op})$ and *L*-sobriety of a \mathbf{C} - \mathcal{M} -*L*-space *W* refers to $\eta_W \in Iso(\mathbf{C}$ - \mathcal{M} -*L*- \mathbf{Top}). The equivalence $\mathbf{SPA}(\mathbf{C})^{op} \sim \mathbf{SOBTop}(\mathbf{C})$ is the central result of the second approach, named as "Fundamental Categorical Duality Theorem", and produces many existing and new dualities [2, 3, 4, 5].

2 Relations Between Two Approaches

As a comparison of the two approaches, the former one is more familiar, and has been utilized by several authors [1, 6, 7], while the latter one has been used only by this author. Although the two approaches are primarily different from each other, we aim in this study to ascertain how they are interrelated. We will particularly show that for categories \mathbf{C} and \mathbf{D} with the properties fulfilling the requirements in both approaches, there exists an adjunction

$$S^* \dashv T^* : \mathbf{SOBTop} (\mathbf{C}) \to \mathbf{D},$$

and be interested in the situation whenever this adjunction turns into an equivalence. We also wish to give applications of the presented results to the categories of certain types of universal algebras (possibly with infinitary operations), e.g., the category of sup-lattices with morphisms all sup-preserving maps [9].

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