A Loomis-Sikorski theorem and functional calculus for a
generalized Hermitian algebra

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This contribution is based on the joint work with David J. Foulis and Anna Jenčová [6].

Generalized Hermitian (GH-) algebras, which were introduced in [9] incorporate several
important algebraic and order theoretic structures including effect algebras [8], MV-algebras
[4], orthomodular lattices [10], Boolean algebras [14], and Jordan algebras [12]. Apart from their
intrinsic interest, all of the latter structures host mathematical models for quantum-mechanical
notions such as observables, states, properties, and experimentally testable propositions [5, 15]
and thus are pertinent in regard to the quantum-mechanical theory of measurement [2].

It turns out that GH-algebras are special cases of the more general synaptic algebras intro-
duced in [7]. Thus, in this paper, it will be convenient for us to treat GH-algebras as special
kinds of synaptic algebras . In most of the paper, we focus on commutative GH-algebras. A
commutative GH-algebra $A$ can be shown to be isomorphic to a lattice ordered Banach algebra
$C(X, \mathbb{R})$, under pointwise operations and partial order, of all continuous real-valued functions
on a basically disconnected compact Hausdorff space $X$.

As indicated by the title, one of our purposes in this paper is to formulate and prove an
analogue for commutative GH-algebras of the classical Loomis-Sikorski representation theorem
for Boolean $\sigma$-algebras [11, 14], and its extension for $\sigma$ MV-algebras and Dedekind $\sigma$-complete
$\ell$-groups [1, 3, 13].

A real observable $\xi$ for a physical system $S$ is understood to be a quantity that can be
experimentally measured, and that when measured yields a result in a specified set $\mathbb{R}_\xi$ of real
numbers. A state $\rho$ for $S$ assigns to $\xi$ an expectation, i.e., the long-run average value of a
sequence of independent measurements of $\xi$ in state $\rho$. If $f$ is a function defined on $\mathbb{R}_\xi$, then
$f(\xi)$ is defined to be the observable that is measured by measuring $\xi$ to obtain, say, the result
$\lambda \in \mathbb{R}_\xi$, and then regarding the result of this measurement of $f(\xi)$ to be $f(\lambda)$.

We use our Loomis-Sikorski theorem to show that each element $a$ in a GH-algebra $A$ corre-
sponds to a real observable $\xi_a$. Moreover, we obtain an integral formula for the expectation of
the observable $\xi_a$ in state $\rho$, and we provide a continuous functional calculus for $A$.

References


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