A Loomis-Sikorski theorem and functional calculus for a generalized Hermitian algebra

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This contribution is based on the joint work with David J. Foulis and Anna Jenčová [6].

Generalized Hermitian (GH-) algebras, which were introduced in [9] incorporate several important algebraic and order theoretic structures including effect algebras [8], MV-algebras [4], orthomodular lattices [10], Boolean algebras [14], and Jordan algebras [12]. Apart from their intrinsic interest, all of the latter structures host mathematical models for quantum-mechanical notions such as observables, states, properties, and experimentally testable propositions [5, 15] and thus are pertinent in regard to the quantum-mechanical theory of measurement [2].

It turns out that GH-algebras are special cases of the more general synaptic algebras introduced in [7]. Thus, in this paper, it will be convenient for us to treat GH-algebras as special kinds of synaptic algebras . In most of the paper, we focus on commutative GH-algebras. A commutative GH-algebra A can be shown to be isomorphic to a lattice ordered Banach algebra $C(X,\mathbb{R})$, under pointwise operations and partial order, of all continuous real-valued functions on a basically disconnected compact Hausdorff space X.

As indicated by the title, one of our purposes in this paper is to formulate and prove an analogue for commutative GH-algebras of the classical Loomis-Sikorski representation theorem for Boolean σ-algebras [11, 14], and its extension for σ MV-algebras and Dedekind σ-complete ℓ-groups [1, 3, 13].

A real observable ξ for a physical system $\mathcal{S}$ is understood to be a quantity that can be experimentally measured, and that when measured yields a result in a specified set $\mathbb{R}_\xi$ of real numbers. A state ρ for $\mathcal{S}$ assigns to ξ an expectation, i.e., the long-run average value of a sequence of independent measurements of ξ in state ρ. If f is a function defined on $\mathbb{R}_\xi$, then $f(\xi)$ is defined to be the observable that is measured by measuring ξ to obtain, say, the result $\lambda \in \mathbb{R}_\xi$, and then regarding the result of this measurement of $f(\xi)$ to be $f(\lambda)$.

We use our Loomis-Sikorski theorem to show that each element a in a GH-algebra A corresponds to a real observable ξ_a. Moreover, we obtain an integral formula for the expectation of the observable ξ_a in state ρ,and we provide a continuous functional calculus for A.

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