Finite MTL-algebras

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Abstract

We obtain a duality between the category of finite MTL-algebras and the category of finite Labeled Trees. In addition we proof that the forest product of MTL-algebras is essentialy a sheaf of MTL-chains over an Alexandrov space.

MTL-logic was introduced by Esteva and Godo in [5] as the basic fuzzy logic of left-continuous t-norms. Furthermore, a new class of algebras was defined, the variety of MTL-algebras. This variety constitutes an equivalent algebraic semantics for MTL-logic. MTL-algebras are essentially integral commutative residuated lattices with bottom satisfying the prelinearity equation:

$$(x \to y) \vee (y \to x) \approx 1$$

We call $f\mathcal{MTL}$ to the algebraic category of finite MTL-algebras.

A totally ordered MTL-algebra (MTL-chain) is archimedean if for every $x \leq y < 1$, there exists $n \in \mathbb{N}$ such that $y^n \leq x$.

A *forest* is a poset X such that for every $a \in X$ the set

$$\downarrow a = \{x \in X \mid x \leq a\}$$

is a totally ordered subset of X. A *p-morphism* is a morphism of posets $f : X \to Y$ satisfying the following property: given $x \in X$ and $y \in Y$ such that $y \leq f(x)$ there exists $z \in X$ such that $z \leq x$ and $f(z) = y$. Let $fa\mathcal{MTL}$ be the algebraic category of finite archimedean MTL-algebras and $fa\mathcal{MTL}c$ the full subcategory of finite archimedean MTL-chains. Let $\mathfrak{C}$ be its skeleton. A *labeled forest* is a function $l : F \to \mathfrak{C}$, where F is a forest. Consider two labeled forests $l : F \to \mathfrak{C}$ and $m : G \to \mathfrak{C}$. A morphism $l \to m$ is a pair $(\varphi, \mathcal{F})$ such that $\varphi : F \to G$ is a p-morphism and $\mathcal{F} = \{f_x\}_{x\in F}$ is a family of morphisms $f_x : m\varphi(x) \to l(x)$ of MTL algebras. We call $f\mathcal{LF}$ to the category of labeled forests and their morphisms.

Definition 1. Let $\mathbf{F} = (F, \leq)$ a forest and let $\{\mathbf{M}_i\}_{i\in\mathbf{F}}$ a collection of MTL-chains such that, up to isomorphism, all they share the same neutral element 1 and the same minimum element 0. If $(\bigcup_{i\in\mathbf{F}})^F$ denotes the set of functions $h : F \to \bigcup_{i\in\mathbf{F}} \mathbf{M}_i$ such that $h(i) \in \mathbf{M}_i$ for all $i \in \mathbf{F}$, the *forest product* $\bigotimes_{i\in\mathbf{F}} A_i$ is the algebra $\mathbf{M}$ defined as follows:

(1) The elements of $\mathbf{M}$ are the $h \in \left(\bigcup_{i\in\mathbf{F}} \mathbf{M}_i\right)^F$ such that, for all $i \in \mathbf{F}$ if $h(i) \neq 0_i$ then for all $j < i$, $h(j) = 1$.

(2) The monoid operation and the lattice operations are defined pointwise.

(3) The residual is defined as follows:

$$(h \to g)(i) = \begin{cases} h(i) \to_i g(i), & \text{if for all } j < i,\ h(j) \leq_j g(j) \\ \\ 0_i & \textit{otherwise} \end{cases}$$

where de subscript $_i$ denotes the realization of operations and of order in $\mathbf{M}_i$.

In every poset $\mathbf{P}$ the collection $\mathcal{D}(\mathbf{P})$ of lower sets of $\mathbf{P}$ defines a topology over P called the *Alexandrov topology* on $\mathbf{P}$. Let $\mathbf{Shv}(\mathcal{D}(\mathbf{P}))$ the category of sheaves over $\mathcal{D}(\mathbf{P})$.

Lemma 1. Let $\mathbf{F}$ a forest and $\{\mathbf{M}_i\}_{i\in\mathbf{F}}$ a collection of MTL-chains as in Definition 1. Then, the assignment $\mathcal{P}:\mathcal{D}(\mathbf{F})^{op}\to\mathbf{Set}$, $\mathcal{P}(U)=\bigotimes_{i\in\mathbf{U}}A_i$ is a sheaf of MTL-algebras in $\mathbf{Shv}(\mathcal{D}(\mathbf{P}))$. Moreover, $\mathcal{P}$ is a sheaf of MTL-chains in $\mathbf{Shv}(\mathcal{D}(\mathbf{P}))$.

Let M be a finite MTL-algebra. A submultiplicative monoid F of M is called a filter if is an upset respect to the order of M. A filter F of M is *prime* if $0\notin F$ and $x\vee y\in F$ entails $x\in F$ or $y\in F$, for every $x,y\in M$. The set of prime filters of a MTL-algebra M ordered by the inclusion will be noted as $\mathrm{Spec}(M)$. Let $\mathcal{I}(M)$ be the poset of idempotent elements of M; $\mathcal{J}(\mathcal{I}(M))^*$ the subposet of non zero join irreducible elements of $\mathcal{I}(M)$ and $m(M)$ the set of minimal elements of $\mathcal{J}(\mathcal{I}(M))^*$. Let $e\in\mathcal{J}(\mathcal{I}(M))^*$ and a_e the smallest $a\in m(M)$ such that $a\leq e$.

Lemma 2. For every finite MTL-algebra M the following holds:

i) $\mathcal{J}(\mathcal{I}(M))^*$ is a finite forest.

ii) For every $e\in\mathcal{J}(\mathcal{I}(M))^*$, $M/\uparrow a_e$ is a finite archimedean MTL-chain, so the function $l_M:\mathcal{J}(\mathcal{I}(M))^*\to\mathfrak{C}$ defined as $l_M(e)=M/\uparrow a_e$ becomes a finite labeled forest.

In this work we pretend to transform the results obtained in Lemmas 1 and 2 in functorial assignments in order to obtain a categorical equivalence between the categories $f\mathcal{LF}$ and $f\mathcal{MTL}$.

It is worth to mention that from the well known equivalence between the topos of sheaves over a topological space X and the topos of local homeos over X the Lemma 1 can be stated as: *The forest product of MTL algebras is isomorphic to the algebra of global sections of a bundle over an Alexandrov space.*

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