Finite MTL-algebras

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Abstract

We obtain a duality between the category of finite MTL-algebras and the category of finite Labeled Trees. In addition we proof that the forest product of MTL-algebras is essentially a sheaf of MTL-chains over an Alexandrov space.

MTL-logic was introduced by Esteva and Godo in [5] as the basic fuzzy logic of left-continuous t-norms. Furthermore, a new class of algebras was defined, the variety of MTL-algebras. This variety constitutes an equivalent algebraic semantics for MTL-logic. MTL-algebras are essentially integral commutative residuated lattices with bottom satisfying the prelinearity equation:

\[(x \rightarrow y) \lor (y \rightarrow x) \approx 1\]

We call \(f_{\text{MTL}}\) to the algebraic category of finite MTL-algebras.

A totally ordered MTL-algebra (MTL-chain) is archimedean if for every \(x \leq y < 1\), there exists \(n \in \mathbb{N}\) such that \(y^n \leq x\).

A forest is a poset \(X\) such that for every \(a \in X\) the set \(\downarrow a = \{x \in X \mid x \leq a\}\) is a totally ordered subset of \(X\). A \(p\)-morphism is a morphism of posets \(f : X \to Y\) satisfying the following property: given \(x \in X\) and \(y \in Y\) such that \(y \leq f(x)\) there exists \(z \in X\) such that \(z \leq x\) and \(f(z) = y\). Let \(f_{\text{aMTL}}\) be the algebraic category of finite archimedean MTL-algebras and \(f_{\text{aMTLc}}\) the full subcategory of finite archimedean MTL-chains. Let \(\mathfrak{C}\) be its skeleton. A labeled forest is a function \(l : F \to \mathfrak{C}\), where \(F\) is a forest. Consider two labeled forests \(l, m : F \to \mathfrak{C}\) and \(F = \{f_x\}_{x \in F}\) is a family of morphisms \(f_x : m \varphi(x) \to l(x)\) of MTL algebras. We call \(f_{\mathfrak{L}F}\) to the category of labeled forests and their morphisms.

Definition 1. Let \(F = (F, \leq)\) a forest and let \(\{M_i\}_{i \in F}\) a collection of MTL-chains such that, up to isomorphism, all they share the same neutral element 1 and the same minimum element 0. If \((\bigcup_{i \in F})^F\) denotes the set of functions \(h : F \to \bigcup_{i \in F} M_i\) such that \(h(i) \in M_i\) for all \(i \in F\), the forest product \(\bigotimes_{i \in F} A_i\) is the algebra \(M\) defined as follows:

1. The elements of \(M\) are the \(h \in (\bigcup_{i \in F} M_i)^F\) such that, for all \(i \in F\) if \(h(i) \neq 0\), then for all \(j < i, h(j) = 1\).

2. The monoid operation and the lattice operations are defined pointwise.

3. The residual is defined as follows:

\[
(h \rightarrow g)(i) = \begin{cases}
  h(i) \rightarrow_i g(i), & \text{if for all } j < i, h(j) \leq_j g(j) \\
  0, & \text{otherwise}
\end{cases}
\]
where de subscript 1 denotes the realization of operations and of order in Mi.

In every poset P the collection D(P) of lower sets of P defines a topology over P called the Alexandrov topology on P. Let Shv(D(P)) the category of sheaves over D(P).

**Lemma 1.** Let F be a forest and {Mi}i∈F a collection of MTL-chains as in Definition 1. Then, the assignment P : D(F)op → Set, P(U) = ⊓i∈U Ai is a sheaf of MTL-algebras in Shv(D(P)). Moreover, P is a sheaf of MTL-chains in Shv(D(P)).

Let M be a finite MTL-algebra. A submultiplicative monoid F of M is called a filter if is an upset respect to the order of M. A filter F of M is prime if 0 ⋚ F and x ∨ y ∈ F, for every x, y ∈ M. The set of prime filters of a MTL-algebra M ordered by the inclusion will be noted as Spec(M). Let I(M) be the poset of idempotent elements of M; J(M)− the subposet of non zero join irreducible elements of I(M) and m(M) the set of minimal elements of J(M). Let e ∈ J(M)− and a the smallest a ∈ m(M) such that a ≤ e.

**Lemma 2.** For every finite MTL-algebra M the following holds:

i) J(M)− is a finite forest.

ii) For every e ∈ J(M)−, M/↑ae is a finite archimedean MTL-chain, so the function lM : J(M)− → C defined as lM(e) = M/↑ae becomes a finite labeled forest.

In this work we pretend to transform the results obtained in Lemmas 1 and 2 in functorial assignments in order to obtain a categorical equivalence between the categories fLF and fMTL.

It is worth to mention that from the well known equivalence between the topos of sheaves over a topological space X and the topos of local homeos over X the Lemma 1 can be stated as: The forest product of MTL algebras is isomorphic to the algebra of global sections of a bundle over an Alexandrov space.

**References**


