Axiomatizing a Reflexive Real-valued Modal Logic

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Many-valued modal logics extend the Kripke frame setting of classical modal logic with a many-valued semantics at each world to model modal notions such as necessity, belief, and spatio-temporal relations in the presence of uncertainty, possibility, or vagueness (see, e.g., [1, 2, 4]). In [3] a many-valued modal logic defined over serial frames with connectives interpreted locally as abelian group operations over the real numbers was introduced and the completeness of an axiomatization established. In this work we extend this result to reflexive frames, thereby taking a first step towards a more general theory of modal logics based on abelian groups. This logic can be viewed as a modal extension of the multiplicative fragment of abelian logic (see, e.g., [5]) and can be axiomatized by adding an axiom expressing reflexivity to the axiom system provided for the logic in [3]. We give a sound and complete axiom system for this logic, where we prove completeness using both a sequent calculus and a labelled tableau system.

Let us denote by Fm the set of formulas defined inductively over a countably infinite set Var of propositional variables using the binary connective $\to$ and modal connective $\Box$. We define

$$\overline{0} := p_0 \to p_0, \quad \neg\varphi := \varphi \to \overline{0}, \quad \varphi \& \psi := \neg\varphi \to \psi, \quad \text{and} \quad \Diamond\varphi := \neg\Box\neg\varphi,$$

and let $0\varphi = \overline{0}$ and $(n+1)\varphi = \varphi\&(n\varphi)$ for all $n \in \mathbb{N}$.

A *frame* is a pair $\mathfrak{F} = \langle W, R\rangle$ such that W is a non-empty set of *worlds* and $R \subseteq W \times W$ is an *accessibility relation* on W. $\mathfrak{F}$ is called *reflexive* if the accessibility relation is reflexive, that is, if for all $x \in W$, Rxx. A KT($\mathbb{R}$)-*model* is a triple $\mathfrak{M} = \langle W, R, V\rangle$ such that $\langle W, R\rangle$ is a reflexive frame and $V\colon \mathrm{Var} \times W \to [-r, r]$ for some $r \in \mathbb{R}_+$ is a *valuation* that extends to $V\colon \mathrm{Fm} \times W \to \mathbb{R}$ via

$$\begin{aligned} V(\varphi \to \psi, x) &= V(\psi, x) - V(\varphi, x) \\ V(\Box\varphi, x) &= \bigwedge\{V(\varphi, y) : Rxy\}. \end{aligned}$$

A formula $\varphi \in \mathrm{Fm}$ will be called *valid* in a KT($\mathbb{R}$)-model $\mathfrak{M} = \langle W, R, V\rangle$ if $V(\varphi, x) \geq 0$ for all $x \in W$. If φ is valid in all KT($\mathbb{R}$)-models, then φ is said to be KT($\mathbb{R}$)-*valid*, written $\models_{\mathrm{KT}(\mathbb{R})} \varphi$.

The proposed axiom system KT($\mathbb{R}$) for this logic is given in Fig. 1.

(B)	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	
(C)	$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	$\dfrac{\varphi \quad \varphi \to \psi}{\psi}$ (mp)
(I)	$\varphi \to \varphi$	
(A)	$((\varphi \to \psi) \to \psi) \to \varphi$	$\dfrac{\varphi}{\Box\varphi}$ (nec)
(K)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$	
(T)	$\Box\varphi \to \varphi$	$\dfrac{n\varphi}{\varphi}$ (con_n) $\quad (n \geq 2)$
(D_n)	$\Box(n\varphi) \to n\Box\varphi \qquad (n \geq 2)$	

Figure 1: The axiom system KT($\mathbb{R}$)

$$\frac{}{\Delta \Rightarrow \Delta}\ (\text{ID})$$

$$\frac{\Gamma \Rightarrow \Delta \quad \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Sigma, \Delta}\ (\text{MIX}) \qquad \frac{n\Gamma \Rightarrow n\Delta}{\Gamma \Rightarrow \Delta}\ (\text{SC}_n) \quad (n \geq 2)$$

$$\frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ (\to\Rightarrow) \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ (\Rightarrow\to)$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Box\varphi \Rightarrow \Delta}\ (\Box\Rightarrow) \qquad \frac{\Gamma \Rightarrow n[\varphi]}{\Box\Gamma \Rightarrow n[\Box\varphi]}\ (\Box_n) \quad (n \geq 0)$$

Figure 2: The sequent calculus GKT($\mathbb{R}$)

That any formula derivable in this system is KT($\mathbb{R}$)-valid is easily shown. To prove the converse, we first introduce the sequent calculus in Fig. 2, where a sequent $\Gamma \Rightarrow \Delta$ is defined to be an ordered pair of finite multisets of formulas, $k\Gamma$ denotes $\Gamma, \ldots, \Gamma$ (k times), and $\Box\Gamma$ denotes the multiset of boxed formulas $[\Box\varphi : \varphi \in \Gamma]$. A sequent can be translated into a formula via the interpretation (where $\varphi_1 \& \ldots \& \varphi_n = \overline{0}$ for $n = 0$):

$$\mathcal{I}(\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m) := (\varphi_1 \& \ldots \& \varphi_n) \to (\psi_1 \& \ldots \& \psi_m).$$

We then prove that $\Gamma \Rightarrow \Delta$ is derivable in GKT($\mathbb{R}$) if and only if $\mathcal{I}(\Gamma \Rightarrow \Delta)$ is derivable in KT($\mathbb{R}$). Completeness is then established via an intermediate labelled tableau calculus in which derivability is equivalent to KT($\mathbb{R}$)-validity. This tableau calculus reduces the problem of proving completeness to solving linear inequations over $\mathbb{R}$. We hence obtain the main result:

Theorem 1. *The following are equivalent for any formula φ:*

(1) *φ is* KT($\mathbb{R}$)*-valid.*

(2) *φ is derivable in* KT($\mathbb{R}$)

(3) *$\Rightarrow \varphi$ is derivable in* GKT($\mathbb{R}$).

References

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