Two systems of point-free affine geometry

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Our presentation is devoted to two systems of geometry which are point-free, in the sense that the notion of *a point* is absent from their basic notions.

The first system, created by the Polish mathematician Aleksander Śniatycki in [3], is based on the notions of *region*, *parthood* and *half-plane*. Śniatycki describes structures of the form:

$$\langle \mathbb{R}, +, \cdot, -, \mathbb{H}, \mathbf{0}, \mathbf{1} \rangle$$

such that $\mathbb{R}$ is a non-empty set whose elements are called *regions*, $\mathbb{H} \subseteq \mathbb{R}$ is a set whose elements are called *half-planes*, and:

$$\langle \mathbb{R}, +, \cdot, -, \mathbf{0}, \mathbf{1} \rangle \text{ is a complete Boolean algebra.} \tag{H0}$$

The specific half-plane postulates (which we formulate here in an abbreviated form) are:

$$h \in \mathbb{H} \longrightarrow -h \in \mathbb{H}\,, \tag{H1}$$

$$\begin{aligned} x_1, x_2, x_3 \in \mathbb{R}^+ \longrightarrow \big(&\exists_{h \in \mathbb{H}} \forall_{i \in \{1,2,3\}} \left(x_i \cdot h \neq \mathbf{0} \neq x_i \cdot -h \,\vee\right. \\ &\exists_{h_1, h_2, h_3 \in \mathbb{H}} \left(\forall_{i \in \{1,2,3\}}\, x_i \cdot -h_i = \mathbf{0} \,\wedge\right. \\ &((x_1 + x_2) \cdot h_3) + ((x_1 + x_3) \cdot h_2) + ((x_2 + x_3) \cdot h_1) = \mathbf{0})\,, \end{aligned} \tag{H2}$$

$$\forall_{h_1, h_2, h_3 \in \mathbb{H}} \left(h_1 \cdot (h_2 + h_3) = \mathbf{0} \longrightarrow h_2 \cdot -h_3 = \mathbf{0} \vee h_3 \cdot -h_2 = \mathbf{0}\right), \tag{H3}$$

$$\begin{aligned} \forall_{h_1, h_2, h_3, h_4 \in \mathbb{H}} \big(&h_1 \cdot h_2 \cdot ((h_3 \cdot -h_4) + (h_4 \cdot -h_3)) = \mathbf{0} \longrightarrow \\ &(h_3 = h_4) \vee (h_1 \cdot h_2 \cdot h_3 = \mathbf{0}) \vee (h_1 \cdot h_2 \cdot -h_3 = \mathbf{0})\big)\,. \end{aligned} \tag{H4}$$

Śniatycki demonstrates that the standard notions of *line*, *point*, *incidence* relation between lines and points and *betweenness* relation on points are definable in his structures, and that the set of axioms he puts forward is sufficiently strong to prove all the axioms of a system of affine geometry (by which, for the purpose of this talk, we may understand the part of geometry which is expressed by *incidence* and *betweenness* only). In this sense, the theory with axioms (H0)–(H4) may be considered as a system of *point-free* affine geometry.

The second system, which comes from [2], pursues the old idea of Alfred Whitehead's [4] of establishing geometry by combining the mereological notions of *region* and *parthood* with that of *oval* as primitives. Indeed, taking the convex opens subsets of the Cartesian plane as paradigms of oval regions we construct geometry in which the notion of *oval* (treated as a point-free counterpart of the notion of *convex* set) is assumed as basic. Via this (and two other notions, of *region* and *parthood*) we introduce *lines* and *half-planes*, and formulate the following axioms which have very natural geometrical interpretation ($\mathbb{O}$ is the set of ovals, $\mathbb{O}^+$ is the set of non-zero ovals):

$$\langle \mathbb{R}, \leqslant \rangle \text{ is a complete atomless Boolean lattice.} \tag{00}$$

$$\mathbb{O} \text{ is an algebraic closure system in } \langle \mathbb{R}, \leqslant \rangle \text{ containing } \mathbf{0}. \tag{01}$$

$$\mathbb{O}^+ \text{ is dense in } \langle \mathbb{R}^+, \leqslant \rangle\,. \tag{02}$$

$$\text{The sides of a line form a partition of } \mathbf{1}\,. \tag{03}$$

For any $a, b, c \in \mathbb{O}^+$ which are not aligned there is a line which separates a from $\mathrm{hull}(b + c)$. (04)

If distinct lines L_1 and L_2 both cross an oval a, then they split a into at least three parts. (05)

$$\text{No half-plane is part of any stripe or angle.} \tag{06}$$

In the axioms above hull is the closure operator arising from $\mathbb{O}$, alignment of regions may be geometrically interpreted as being crossed by one line (which is a pair of maximal ovals), stripe is the product of two «parallel» half-planes, and angle may be interpreted in the traditional Euclidean way as the intersection of two half-planes which are not «parallel».

About the theory composed of axioms (00)–(06) we prove that its definitional extension with the notion of *half-plane* is strong enough to prove all Śniatycki's axioms, and therefore is suitable for reconstruction of affine geometry. So this theory deserves the name of *point-free* affine geometry as well.

In our talk we would like to:

(i) present geometrical interpretation of (cryptic at first sight) axioms of Śniatycki's,

(ii) describe the steps in construction of our system from [2] and justify our choice of axioms from point of view of pursuing affine geometry,

(iii) and sketch the proof of axioms of Śniatycki's from (00)–(06).

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