The Convolution Algebra

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A relational structure, or frame, $\mathfrak{X} = (X, (R_i)_I)$ is a set X with a family $(R_i)_I$ of relations on X where we assume that R_i is $n_i + 1$-ary. The complex algebra $\mathfrak{X}^+ = (\mathcal{P}(X), (\Diamond_i)_I, (\Box_i)_I)$ of this frame is the algebra consisting of the power set $\mathcal{P}(X)$ of X with n_i-ary operations $\Diamond_i$ and $\Box_i$ for each $i \in I$. We follow the Jónsson and Tarski method of defining $\Diamond_i$ through the relational image of R_i with $\Box_i$ as its dual. When $\mathcal{P}(X)$ is viewed as 2^X, these operations are given by

$$\Diamond_i(f_1, \ldots, f_{n_i})(x) = \bigvee\{f(x_1) \wedge \cdots \wedge f(x_n) : (x_1, \ldots, x_n, x) \in R_i\}$$

$$\Box_i(f_1, \ldots, f_{n_i})(x) = \bigwedge\{f(x_1) \vee \cdots \vee f(x_n) : (x_1, \ldots, x_n, x) \in R_i\}$$

Replacing 2 with a complete lattice L leads to an obvious generalization of this construction to what we call the convolution algebra $L^{\mathfrak{X}}$. The name is given to reflect that the operations $(\Diamond_i)_I$ and $(\Box_i)_I$ are obtained via a form of convolution.

We consider basic properties of this convolution algebra. Among our results, we show that when L is a non-trivial complete Heyting algebra that the operations $\Diamond_i$ are complete operators and that $L^{\mathfrak{X}}$ and $\mathfrak{X}^+$ satisfy the same equations in the signature $\wedge, \vee, 0, 1, (\Diamond_i)_I$. The dual result holds when L is a non-trivial complete dual Heyting algebra. When L is non-trivial, complete, and completely distributive, $L^{\mathfrak{X}}$ and $\mathfrak{X}^+$ satisfy the same equations in $\wedge, \vee, 0, 1, (\Diamond_i)_I, (\Box_i)_I$.

Frames of a given type $\tau' = (n_i + 1)_I$ form a category $\textsc{Frm}_{\tau'}$ with the morphisms being p-morphisms. Then considering the category $\textsc{Lat}$ of complete lattices with morphisms being maps that preserve finite meets and arbitrary joins, and $\textsc{Alg}_\tau$ the category of algebras of type $\tau = (n_i)_I$, there is a bifunctor

$$\textsc{Conv} : \textsc{Lat} \times \textsc{Frm}_{\tau'} \longrightarrow \textsc{Alg}_\tau$$

that is covariant in the first argument and contravariant in the second. Here we are considering the restriction to the $(\Diamond_i)_I$ fragment. Modifications to the morphisms of $\textsc{Lat}$ provide versions for $(\Box_i)_I$ fragment, and to the full language. Various results are shown related to the behavior of this bifunctor with respect to one-one and onto maps, and with respect to products and coproducts in its two components.

Several examples are considered. These include monadic Heyting algebras; versions of intuitionistic relation algebras obtained from $H^{\mathfrak{G}}$ where H is a complete Heyting algebra and $\mathfrak{G}$ is a group; and the convolution algebra $\mathrm{I}^{\mathfrak{I}}$ where I is the real unit interval and the relational structure $\mathfrak{I} = (\mathrm{I}, \wedge, \vee, 0, 1, \neg, \triangle, \triangledown)$ consists of I with its max and min operations, bounds, negation, and a t-norm $\triangle$ and co-norm $\triangledown$. This algebra $\mathrm{I}^{\mathfrak{I}}$ is the truth value object used in type-2 fuzzy sets.

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