Projective WS5-Algebras

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The logic WS5 plays an important role in extending Glivenko’s Theorem to MIPC (see [2]). The algebraic models for WS5 are monadic Heyting algebras in which the open elements form a Boolean algebra. We study the variety \( \mathcal{M} \) of such algebras from the standpoint of projectivity. We give a description of \( F_{\mathcal{M}}(1) \), and we prove a criterion of projectivity of finitely-presented algebra from any of subvarieties of \( \mathcal{M} \).

Free Single-Generated Algebra

An algebra \( \langle A; \land, \lor, \to, 0, 1, \square \rangle \), where \( \langle A; \land, \lor, \to, 0, 1 \rangle \) is a Heyting algebra and \( \square \) satisfies the following identities:

\[
\begin{align*}
(M0) \quad & \square 1 \approx 1; \\
(M1) \quad & \square x \to x \approx 1; \\
(M2) \quad & \square(x \to y) \to (\square x \to \square y) \approx 1; \\
(M3) \quad & \square x \to \square \square x \approx 1; \\
(M4) \quad & \lnot \square \lnot \square x \approx \lnot \square x.
\end{align*}
\]

is called an \( m \)-algebra. It is clear that the set of all \( m \)-algebras forms a variety that we denote by \( \mathcal{M} \). All necessary information about monadic Heyting algebras (including \( m \)-algebras) can be found in [1]. An element \( a \) of an \( m \)-algebra is open, if \( a = \square a \). Recall that an \( m \)-algebra is subdirectly irreducible (s.i. for short), if it has exactly two open elements: 0 and 1.

For any element \( a \) of any \( m \)-algebra \( A \), we define the degrees of \( a \) as follows: \( a^0 := 0 \), \( a^1 := \lnot a \), \( a^2 := a \) and for all \( k \geq 0 \) \( a^{2k+3} := a^{2k+1} \to a^{2k} \), \( a^{2k+4} := a^{2k+2} \lor a^{2k+3} \), and we let \( a^\omega := 1 \).

For \( n > 1 \) we denote by \( Z_n \) a single-generated s.i. \( m \)-algebra of cardinality \( n \). The Heyting reduct of \( Z_n \) (H-reduct for short) is a single-generated Heyting algebra with \( n \) elements in which \( \square 1 = 1 \) and \( \square a = 0 \) for all \( a < 1 \). Every algebra \( Z_n \) consists of degrees of its generator that we denote by \( g_n \). \( Z_2 \) is a two-element \( m \)-algebra with generator \( g_2 = 0 \), while by \( Z_1 \) we denote a two-element \( m \)-algebra with generator \( g_1 = 1 \).

Let

\[
P = \prod_{i > 0} Z_i
\]

and \( Z \) be a subalgebra of \( P \) generated by element \( g = (g_1, g_2, \ldots) \),

that is, by the element \( g \) such that \( \pi_i(g) = g_i, i > 0 \), where \( \pi_i \) is a \( i \)-th projection.

**Proposition 1.** \( Z \) is isomorphic to \( F_{\mathcal{M}}(1) \).

An element \( a \in P \) is called leveled, if there are \( 0 < k < \omega \) and \( 0 < m < \omega \) such that \( \pi_j(a) = g_j^m \) for all \( j \geq k \). Let \( L \) be a set of all leveled elements of \( P \). The following theorem gives a convenient intrinsic description of \( F_{\mathcal{M}}(1) \).

**Theorem 2.** \( L = Z \), hence \( F_{\mathcal{M}}(1) \) is isomorphic to a subalgebra of \( P \) consisting of all leveled elements.

As one can see from the following corollary, the structure of \( F_{\mathcal{M}}(1) \) is much more complex than the structure of free single-generated Heyting algebra.
Corollary 3. The following holds

(a) $H$-reduct of $F_M(1)$ is not finitely generated as Heyting algebra;
(b) $F_M(1)$ contains infinite ascending and descending chains of open elements;
(c) $F_M(1)$ is atomic and it has infinite set of atoms;
(d) $Z_2$ is the only s.i. subalgebra of $F_M(1)$.

Projective Algebras

In the following theorem we use the notations from [1]: $\varphi(A)$ denotes the $H$-reduct of $A$, $\psi(A)$ denotes a relatively complete subalgebra of $\varphi(A)$ defining modal operations, and $\psi(V) = \{\psi(A) : A \in V\}$.

Theorem 4. (comp. [3, Corollary 5.5]) Let $V \subseteq MHA$ be a variety of monadic Heyting algebras. If $A \in V$ is such an algebra that $\varphi(A) = \psi(A)$ and algebra $\psi(A)$ is projective in $\psi(V)$, then $A$ is projective in $V$.

Corollary 5. If $A$ is at most countable m-algebra and each element of $A$ is open, then $A$ is projective in $M$.

Proposition 6. Each projective in $MHA$ algebra has $Z_2$ as a homomorphic image.

Let $V$ be a variety of m-algebras and $A \in V$. Then $A \in V$ is finitely presented in $V$ if $A \cong F_V(n)/\theta$ for some $n$, where $\theta$ is a principal congruence on $F_V(n)$.

The following theorem extends the criterion of projectivity [4, Theorem 5.2] from finite to finitely-presented m-algebras.

Theorem 7. Let $V$ be a variety of m-algebras and $A \in V$ be finitely presented in $V$. Then $A$ is projective in $V$ if and only if $Z_2$ is a homomorphic image of $A$.

Corollary 8. Let $V$ be a variety of m-algebras. Then every finitely presented subalgebra of $F_V(\omega)$ is projective in $V$. In particular, every finite subalgebra of $F_V(\omega)$ is projective.

Corollary 9. Let $V$ be a variety of m-algebras and $A \in V$ be given by defining relation $t(x_1, \ldots, x_n) = 1$. Then $A$ is projective in $V$ if and only if the term $t$ is satisfiable in $Z_2$.

Corollary 10. Let $V$ be a variety of m-algebras. Then the problem whether a given finite set of equations defines in $V$ a projective finitely presented algebra is decidable.

Corollary 11. $Z_2$ is the only projective s.i. m-algebra.

Corollary 12. Let $V$ be a variety of m-algebras. Then the problem whether a given finite set of equations defines in $V$ a projective finitely presented algebra is decidable.

Theorem 13. For every finitely generated m-algebra $A$ the following is equivalent

(a) $A$ has $Z_2$ as a homomorphic image;
(b) $A$ does not contain an element $a$ such that $\Box a = \Box \neg a$;
(c) quasi-identity $\rho := (\neg \Box x \land \neg \Box \neg x) \equiv 1 \Rightarrow 0$ holds on $A$.

Corollary 14. The quasivariety $Q$ defined by quasi-identity $\rho$ is primitive and $Q$ contains every primitive quasivariety of $m$-algebras as a subquasivariety.
References