Projective WS5-Algebras

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The logic WS5 plays an important role in extending Glivenko's Theorem to MIPC (see [2]). The algebraic models for WS5 are monadic Heyting algebras in which the open elements form a Boolean algebra. We study the variety $\mathcal{M}$ of such algebras from the standpoint of projectivity. We give a description of $\mathbf{F}_{\mathcal{M}}(1)$, and we prove a criterion of projectivity of finitely-presented algebra from any of subvarieties of $\mathcal{M}$.

Free Single-Generated Algebra

An algebra $\langle \mathsf{A}; \wedge, \vee, \to, \mathbf{0}, \mathbf{1}, \Box \rangle$, where $\langle \mathsf{A}; \wedge, \vee, \to, \mathbf{0}, \mathbf{1} \rangle$ is a Heyting algebra and $\Box$ satisfies the following identities:

$$\begin{array}{ll} (M0) & \Box \mathbf{1} \approx \mathbf{1}; \\ (M1) & \Box x \to x \approx \mathbf{1}; \\ (M2) & \Box(x \to y) \to (\Box x \to \Box y) \approx \mathbf{1}; \\ (M3) & \Box x \to \Box \Box x \approx \mathbf{1}; \\ (M4) & \neg \Box \neg \Box x \approx \Box x. \end{array}$$

is called an *m-algebra*. It is clear that the set of all m-algebras forms a variety that we denote by $\mathcal{M}$. All necessary information about monadic Heyting algebras (including m-algebras) can be found in [1]. An element a of an m-algebra is *open*, if $\mathsf{a} = \Box\mathsf{a}$. Recall that an m-algebra is subdirectly irreducible (s.i. for short), if it has exactly two open elements: $\mathbf{0}$ and $\mathbf{1}$.

For any element a of any m-algebra $\mathbf{A}$, we define the degrees of a as follows: $\mathsf{a}^0 := \mathbf{0}$, $\mathsf{a}^1 := \neg\mathsf{a}$, $\mathsf{a}^2 := \mathsf{a}$ and for all $k \geq 0$ $\mathsf{a}^{2k+3} := \mathsf{a}^{2k+1} \to \mathsf{a}^{2k}$, $\mathsf{a}^{2k+4} := \mathsf{a}^{2k+1} \vee \mathsf{a}^{2k+2}$, and we let $\mathsf{a}^{\omega} := \mathbf{1}$.

For $n > 1$ we denote by $\mathbf{Z}_n$ a single-generated s.i. m-algebra of cardinality n. The Heyting reduct of $\mathbf{Z}_n$ (H-reduct for short) is a single-generated Heyting algebra with n elements in which $\Box\mathbf{1} = \mathbf{1}$ and $\Box\mathsf{a} = \mathbf{0}$ for all $\mathsf{a} < \mathbf{1}$. Every algebra $\mathbf{Z}_n$ consists of degrees of its generator that we denote by g_n. $\mathbf{Z}_2$ is a two-element m-algebra with generator $\mathsf{g}_2 = \mathbf{0}$, while by $\mathbf{Z}_1$ we denote a two-element m-algebra with generator $\mathsf{g}_1 = \mathbf{1}$.

Let

$$\mathbf{P} = \prod_{i>0} \mathbf{Z}_i \text{ and } \mathbf{Z} \text{ be a subalgebra of } \mathbf{P} \text{ generated by element } \mathsf{g} = (\mathsf{g}_1, \mathsf{g}_2, \dots),$$

that is, by the element g such that $\pi_i(\mathsf{g}) = \mathsf{g}_i, i > 0$, where π_i is a i-th projection.

Proposition 1. $\mathbf{Z}$ *is isomorphic to* $\mathbf{F}_{\mathcal{M}}(1)$.

An element $\mathsf{a} \in \mathbf{P}$ is called *leveled*, if there are $0 < k < \omega$ and $0 < m \leq \omega$ such that $\pi_j(\mathsf{a}) = \mathsf{g}_j^m$ for all $j \geq k$. Let $\mathbf{L}$ be a set of all leveled elements of $\mathbf{P}$. The following theorem gives a convenient intrinsic description of $\mathbf{F}_{\mathcal{M}}(1)$.

Theorem 2. $\mathbf{L} = \mathbf{Z}$, *hence* $\mathbf{F}_{\mathcal{M}}(1)$ *is isomorphic to a subalgebra of* $\mathbf{P}$ *consisting of all leveled elements.*

As one can see from the following corollary, the structure of $\mathbf{F}_{\mathcal{M}}(1)$ is much more complex than the structure of free single-generated Heyting algebra.

Corollary 3. *The following holds*

(a) *H-reduct of* $\mathbf{F}_{\mathcal{M}}(1)$ *is not finitely generated as Heyting algebra;*
(b) $\mathbf{F}_{\mathcal{M}}(1)$ *contains infinite ascending and descending chains of open elements;*
(c) $\mathbf{F}_{\mathcal{M}}(1)$ *is atomic and it has infinite set of atoms;*
(d) $\mathbf{Z}_2$ *is the only s.i. subalgebra of* $\mathbf{F}_{\mathcal{M}}(1)$.

Projective Algebras

In the following theorem we use the notations from [1]: $\varphi(\mathbf{A})$ denotes the H-reduct of $\mathbf{A}$, $\psi(\mathbf{A})$ denotes a relatively complete subalgebra of $\varphi(\mathbf{A})$ defining modal operations, and $\psi(\mathcal{V}) = \{\psi(\mathbf{A}) \mid \mathbf{A} \in \mathcal{V}\}$.

Theorem 4. *(comp. [3, Corollary 5.5]) Let* $\mathcal{V} \subseteq \mathsf{MHA}$ *be a variety of monadic Heyting algebras. If* $\mathbf{A} \in \mathcal{V}$ *is such an algebra that* $\varphi(\mathbf{A}) = \psi(\mathbf{A})$ *and algebra* $\psi(\mathbf{A})$ *is projective in* $\psi(\mathcal{V})$*, then* $\mathbf{A}$ *is projective in* $\mathcal{V}$.

Corollary 5. *If* $\mathbf{A}$ *is at most countable m-algebra and each element of* $\mathbf{A}$ *is open, then* $\mathbf{A}$ *is projective in* $\mathcal{M}$.

Proposition 6. *Each projective in* MHA *algebra has* $\mathbf{Z}_2$ *as a homomorphic image.*

Let $\mathcal{V}$ be a variety of m-algebras and $\mathbf{A} \in \mathcal{V}$. Then $\mathbf{A} \in \mathcal{V}$ is *finitely presented* in $\mathcal{V}$ if $\mathbf{A} \cong \mathbf{F}_{\mathcal{V}}(n)/\theta$ for some n, where θ is a principal congruence on $\mathbf{F}_{\mathcal{V}}(n)$.

The following theorem extends the criterion of projectivity [4, Theorem 5.2] from finite to finitely-presented m-algebras.

Theorem 7. *Let* $\mathcal{V}$ *be a variety of m-algebras and* $\mathbf{A} \in \mathcal{V}$ *be finitely presented in* $\mathcal{V}$*. Then* $\mathbf{A}$ *is projective in* $\mathcal{V}$ *if and only if* $\mathbf{Z}_2$ *is a homomorphic image of* $\mathbf{A}$.

Corollary 8. *Let* $\mathcal{V}$ *be a variety of m-algebras. Then every finitely presented subalgebra of* $\mathbf{F}_{\mathcal{V}}(\omega)$ *is projective in* $\mathcal{V}$*. In particular, every finite subalgebra of* $\mathbf{F}_{\mathcal{V}}(\omega)$ *is projective.*

Corollary 9. *Let* $\mathcal{V}$ *be a variety of m-algebras and* $\mathbf{A} \in \mathcal{V}$ *be given by defining relation* $t(x_1, \dots, x_n) = \mathbf{1}$*. Then* $\mathbf{A}$ *is projective in* $\mathcal{V}$ *if and only if the term* t *is satisfiable in* $\mathbf{Z}_2$.

Corollary 10. *Let* $\mathcal{V}$ *be a variety of m-algebras. Then the problem whether a given finite set of equations defines in* $\mathcal{V}$ *a projective finitely presented algebra is decidable.*

Corollary 11. $\mathbf{Z}_2$ *is the only projective s.i. m-algebra.*

Corollary 12. *Let* $\mathcal{V}$ *be a variety of m-algebras. Then the problem whether a given finite set of equations defines in* $\mathcal{V}$ *a projective finitely presented algebra is decidable.*

Theorem 13. *For every finitely generated m-algebra* $\mathbf{A}$ *the following is equivalent*

(a) $\mathbf{A}$ *has* $\mathbf{Z_2}$ *as a homomorphic image;*
(b) $\mathbf{A}$ *does not contain an element* a *such that* $\Box \mathsf{a} = \Box\neg\mathsf{a}$;
(c) *quasi-identity* $\rho \coloneqq (\neg \Box x \wedge \neg \Box \neg x) \approx \mathbf{1} \Rightarrow \mathbf{0}$ *holds on* $\mathbf{A}$.

Corollary 14. *The quasivariety* $\mathcal{Q}$ *defined by quasi-identity* ρ *is primitive and* $\mathcal{Q}$ *contains every primitive quasivariety of m-algebras as a subquasivariety.*

References

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