Sasaki projections and related operations

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Orthomodular lattices were introduced as event structures of quantum mechanics. They admit the modeling of events which are not simultaneously observable. They are not distributive. Therefore the computation in orthomodular lattices is much more advanced than in Boolean algebras. The use of the lattice operations was mostly exhausted and there seems not to be much space for breaking results concerning their properties. Therefore we considered other operations in our previous work. We have shown that there are no other useful associative operations. Among the non-associative ones, Sasaki projection (and its dual) satisfy the most equations which are weakenings of associativity. We collected many of the known properties of Sasaki projection, added new ones, and concentrated on the question of which elements of an orthomodular lattice have a common complement. Here we extend these results.

An *orthomodular lattice* (abbr. OML) is a bounded lattice with an antitone involution $^\perp$ (*orthocomplementation*) satisfying $x \vee x^\perp = \mathbf{1}$, $x \wedge x^\perp = \mathbf{0}$, and $x \leq y \implies y = x \vee (x^\perp \wedge y)$ (*orthomodular law*). A prototypical example of an orthomodular lattice is the lattice of all closed linear subspaces of a Hilbert space with $x^\perp$ being the closure of $\{\boldsymbol{u} \mid \boldsymbol{u} \perp \boldsymbol{v} \text{ for all } \boldsymbol{v} \in x\}$. Without the use of the inner product, only some properties of subspaces can be expressed in algebraic terms of OMLs. Sasaki [11] showed that the orthogonal projection of a subspace y to a subspace x can be expressed without the use of the inner product as the *Sasaki projection* ϕ_x,

$$\phi_x(y) = x \wedge (x^\perp \vee y)\,.$$

Sasaki projections are also studied in [1, 2, 3, 9, 10] and generalized in the context of *synaptic algebras* by D. Foulis and S. Pulmannová in [5].

Throughout this abstract, L denotes an orthomodular lattice and $\Phi(L) = \{\phi_x \mid x \in L\}$. The fundamental observation [10] is that kernels of congruences in L are exactly the subsets I satisfying $\phi_x(y) \in I$ whenever $x \in I$ or $y \in I$. (The meet, $x \wedge y$, does not possess this property.)

Sasaki projections preserve the join [3, 4], i.e., $\phi_x(y \vee z) = \phi_x(y) \vee \phi_x(z)$, therefore they are monotonic. Each monotonic mapping $\theta\colon L \to L$ has a unique *dual*, which is a monotonic mapping $\overline{\theta}\colon L \to L$ defined by

$$\overline{\theta}(y) = (\theta(y^\perp))^\perp\,.$$

The composition of two Sasaki projections, $\phi_x\phi_y$, is a Sasaki projection iff x and y *commute*, i.e., $x = (x \wedge y) \vee (x \wedge y')$. All finite compositions of Sasaki projections on L form a monoid $S(L)$.

For $x_1, \ldots, x_n \in L$, we study the compositions $\xi = \phi_{x_n} \cdots \phi_{x_2}\phi_{x_1}, \xi^* = \phi_{x_1}\phi_{x_2} \cdots \phi_{x_n} \in S(L)$. They form an *adjoint pair*, i.e., $\xi^*(y) = \min\{z \in L \mid \overline{\xi}(z) \geq y\}$, thus each of them uniquely determines the other and the mapping $^*\colon S(L) \to S(L)$ is correctly defined (although the representations of ξ, ξ^* as compositions of Sasaki projections are not unique).

Elements x and y of an OML L are said to be *strongly perspective* if they have a common (relative) complement in the interval $[\mathbf{0}, x \vee y]$. Following [2], we ask when $\xi(\mathbf{1}), \xi^*(\mathbf{1})$ are

strongly perspective. For $n = 2$, this is always the case. We have found a constructive proof for $n = 3$ and a counterexample for $n = 4$ [8]. Chevalier and Pulmannová [2] have given a non-constructive proof for *complete modular* OMLs and arbitrary n; however, a *constructive* proof for $n > 3$ is not known.

Let S be a semigroup with an absorbing element 0 and an involution $^*\colon S \to S$ such that for any $\theta, \eta \in S$, $(\theta\eta)^* = \eta^*\theta^*$. We call S a *Baer *-semigroup* if, for each $\theta \in S$, there is a greatest element, θ', of the right ideal $\{\eta \in S \mid \theta\eta = 0\}$ and $\pi = \theta'$ is a projection, i.e., $\pi = \pi^2 = \pi^*$ [2, 3, 4]. We denote by $P(S)$ the set $\{\theta' \mid \theta \in S\}$.

The theory of Baer *-semigroups can be directly applied to the monoid $S(L)$. Projections of $S(L)$ are exactly the Sasaki projections, $P(S(L)) = \Phi(L)$. The order on $\Phi(L)$ is defined by $\theta \leq \eta \iff \theta\eta = \theta$. For each $\theta \in S(L)$, we define $\theta' := \phi_{\overline{\theta^*}(\mathbf{0})}$. It is the unique element such that for any $\eta \in S(L)$

$$\theta\eta = \phi_{\mathbf{0}} \iff \eta(y) \leq \theta'(y) \text{ for all } y \in L\,. \tag{1}$$

The mapping $'$ is an orthocomplementation which equips $\Phi(L)$ with the structure of an OML. The mapping $\Phi\colon L \to \Phi(L)$, $x \mapsto \phi_x$, is an isomorphism. We derived many of the relevant results using the tools of OML computations.

Chevalier and Pulmannová [2] proved that $\xi^*\xi(\mathbf{1}) = \xi^*(\mathbf{1})$. Their proof heavily used methods of Baer *-semigroups, a direct proof using techniques of OMLs is still not known.

We bring arguments why Sasaki operations form a promissing alternative to lattice operations (join and meet) in the study of orthomodular lattices. Only the lattice operations "satisfy more equations" than Sasaki operations. The potential of using Sasaki operations in the algebraic foundations of orthomodular lattices is still not sufficiently exhausted. Besides, they have a natural physical interpretation.

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