Standard completeness for mianorm-based logics with n-contraction, n-mingle, and n-potency axioms

Eunsuk Yang[1]

Chonbuk National University, Jeonju, Korea
eunsyang@jbnu.ac.kr

1 Introduction

The aim of this paper is to introduce standard completeness results for substructural fuzzy logics based on mianorms (binary monotonic identity aggregation operations on the real unit interval $[0,1]$) with n-contraction, n-mingle, and n-potency axioms. For this, we note that Baldi [1] introduced Wang's $\mathbf{C}_n\mathbf{UL}$ (Uninorm logic **UL** with n-potency) as **UL** with both the n-contraction axiom and the n-mingle axiom. However, micanorm- and mianorm-based logics with each of these axioms have not yet been investigated. Furthermore, we can divide n-contraction, n-mingle, and n-potency axioms into right and left ones in the context of non-commutative logic. Here, we introduce such logic systems and their standard completeness via Yang's construction in the style of Jenei–Montagna (see [3, 4]).

2 Logic systems, Algebras, and Standard completeness

Let φ^n and ${}^n\varphi$ stand for $((...(\varphi\&\varphi)\&\cdots\&\varphi)\&\varphi$, n φ's, and $\varphi\&(\varphi\&\cdots\&(\varphi\&\varphi)...))$, n φ's, respectively. We introduce the following extensions of **MIAL** (Mianorm logic, $=\mathrm{SL}^\ell$).

Definition 1. *Let $2 \leq n$. $\boldsymbol{C^r_n MIAL}$ is $\boldsymbol{MIAL}$ plus (right n-contraction, c^r_n) $\varphi^{n-1} \to \varphi^n$; $\boldsymbol{C^l_n MIAL}$ is $\boldsymbol{MIAL}$ plus (left n-contraction, c^l_n) ${}^{n-1}\varphi \to {}^n\varphi$; $\boldsymbol{M^r_n MIAL}$ is $\boldsymbol{MIAL}$ plus (right n-mingle, m^r_n) $\varphi^n \to \varphi^{n-1}$; $\boldsymbol{M^l_n MIAL}$ is $\boldsymbol{MIAL}$ plus (left n-mingle, m^l_n) ${}^n\varphi \to {}^{n-1}\varphi$; $\boldsymbol{P^r_n MIAL}$ is $\boldsymbol{MIAL}$ plus (right n-potency, p^r_n) $\varphi^{n-1} \leftrightarrow \varphi^n$; and $\boldsymbol{P^l_n MIAL}$ is $\boldsymbol{MIAL}$ plus (left n-potency, p^l_n) ${}^{n-1}\varphi \leftrightarrow {}^n\varphi$.*

Definition 2. *$Ls = \{\boldsymbol{C^r_n MIAL}, \boldsymbol{C^l_n MIAL}, \boldsymbol{M^r_n MIAL}, \boldsymbol{M^l_n MIAL}, \boldsymbol{P^r_n MIAL}, \boldsymbol{P^l_n MIAL}\}$.*

An $\mathcal{A}$-*evaluation* is a function $v : Fm \to \mathcal{A}$ satisfying: $v(\sharp(\varphi_1,\ldots,\varphi_m)) = \sharp^{\mathcal{A}}(v(\varphi_1),\ldots,v(\varphi_m))$, where $\sharp \in \{\to, \rightsquigarrow, \wedge, \vee, \&, \top, \bot, \overline{1}, \overline{0}\}$ and $\sharp^{\mathcal{A}} \in \{\backslash, /, \wedge, \vee, *, \top, \bot, t, f\}$. A formula φ is *valid* in $\mathcal{A}$ if $v(\varphi) \geq t$ for each $\mathcal{A}$-evaluation v. An $\mathcal{A}$-evaluation v is an $\mathcal{A}$-*model* of T if $v(\varphi) \geq t$ for each $\varphi \in T$.

Definition 3. *For $\boldsymbol{L}$ an extension of $\boldsymbol{MIAL}$, a $\boldsymbol{MIAL}$-algebra $\mathcal{A}$ is an $\boldsymbol{L}$-algebra if all axioms of L are valid in $\mathcal{A}$. Especially, for all $x \in A$ and $2 \leq n$, A $\boldsymbol{C^r_n MIAL}$-algebra is a $\boldsymbol{MIAL}$-algebra satisfying ($c^{r\mathcal{A}}_n$) $x^{n-1} \leq x^n$; A $\boldsymbol{C^l_n MIAL}$-algebra is a $\boldsymbol{MIAL}$-algebra satisfying ($c^{l\mathcal{A}}_n$) ${}^{n-1}x \leq {}^n x$; An $\boldsymbol{M^r_n MIAL}$-algebra is a $\boldsymbol{MIAL}$-algebra satisfying ($m^{r\mathcal{A}}_n$) $x^n \leq x^{n-1}$; An $\boldsymbol{M^r_n MIAL}$-algebra is a $\boldsymbol{MIAL}$-algebra satisfying ($m^{l\mathcal{A}}_n$) ${}^n x \leq {}^{n-1}x$; A $\boldsymbol{C^r_n MIAL}$-algebra is a $\boldsymbol{MIAL}$-algebra satisfying ($p^{r\mathcal{A}}_n$) $x^{n-1} = x^n$; A $\boldsymbol{C^l_n MIAL}$-algebra is a $\boldsymbol{MIAL}$-algebra satisfying ($p^{l\mathcal{A}}_n$) ${}^{n-1}x = {}^n x$. For convenience, we call all these algebras* L-algebras.

Theorem 4. (*Strong completeness*) *Let T be a theory over L ($\in Ls$) and φ a formula. $T \vdash_L \varphi$ iff for every linearly ordered L-algebra $\mathcal{A}$ and an $\mathcal{A}$-evaluation v, if v is an $\mathcal{A}$-model of T, then $v(\varphi) \geq t$.*

Proposition 5. *For every finite or countable, linearly ordered L-algebra* $\boldsymbol{A} = (A, \leq_A, \top, \bot, t, f, \wedge, \vee, *, \backslash, /)$*, there is a countable ordered set* X*, a binary operation* $\circ$ *on* X*, and a map* h *from* A *into* X *such that (I)* X *is densely ordered and has a maximum* Max*, a minimum* Min*, and special elements* e *and* ∂*; (II)* $(X, \circ, \preceq, e)$ *is a linearly ordered, monotonic groupoid with unit; (III)* $\circ$ *is conjunctive and left-continuous with respect to (w.r.t.) the order topology on* $(X, \preceq)$*; (IV)* h *is an embedding of the structure* $(A, \leq_A, \top, \bot, t, f, \wedge, \vee, *)$ *into* $(X, \preceq, Max,\ Min, e, \partial, min, max, \circ)$*, and, for all* $m, n \in A$*,* $h(m\backslash n)$ *and* $h(n/m)$ *are the residuated pair of* $h(m)$ *and* $h(n)$ *in* $(X, \preceq, Max,\ Min, e, \partial, max, min, \circ)$*; and (V)* $\circ$ *satisfies right and left* n*-contraction,* n*-mingle, and* n*-potency properties corresponding to* $*$*.*

Theorem 6. (*Strong standard completeness*) *For* $L \in Ls$*,* $T \vdash_L \varphi$ *iff for every standard L-algebra and evaluation* v*, if* $v(\psi) \geq e$ *for all* $\psi \in T$*, then* $v(\varphi) \geq e$*.*

Remark 7. *[(i)]*

1. *For* $L \in Ls$*,* L_e *is L plus (e, exchange)* $(\varphi \& \psi) \to (\psi \& \varphi)$*. By almost the same construction, we can prove standard completeness for* L_e*. But this construction does not work for* L_a*, L plus (a, associativity)* $(\varphi \& \psi) \& \chi \leftrightarrow \varphi \& (\psi \& \chi)$*, since the operation* $\circ$ *for* L_a *does not satisfy associativity (see Theorem 7 (v) in [3]).*

2. *The operation* $\circ$ *in Wang's construction in the style of Jenei–Montagna satisfy associativity (see Theorem 4.3 in [2]). But this construction does not work for* n*-contraction and* n*-potency, for given* $n = 2$*, since* $(m, x) \not\preceq (m, x) \circ (m, x)$ *(see p. 212 in [2]).*

References

[1] Baldi, Paolo. A note on standard completeness for some extensions of uninorm logic, 2014, Soft Computing 18, 1463-1470.

[2] Wang, San-min. Uninorm logic with the n-potency axiom, 2012, Fuzzy Sets and Systems 205, 116-126.

[3] Yang, Eunsuk. Weakening-free, non-associative fuzzy logics: Micanorm-based logics, 2015, Fuzzy Sets and Systems 276, 43-58.

[4] Yang, Eunsuk. Basic substructural core fuzzy logics and their extensions: Mianorm-based logics, 2016, Fuzzy Sets and Systems 301, 1-18.