Undefinability of Standard Sequent Calculi for 3-valued Paraconsistent Logics

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Distinguishing between a "strong sense" and a "weak sense" of propositional connectives when partially defined predicates are present in a language is an idea due to Kleene [12]. Each of these meanings is made explicit by introducing 3-valued truth tables, which have become widely known as *strong Kleene tables* and *weak Kleene tables*. By labelling the elements as $0, n, 1$, the *strong tables* for conjunction, disjunction and negation are displayed below:

$\wedge$	0	n	1
0	0	0	0
n	0	n	n
1	0	n	1

$\vee$	0	n	1
0	0	n	1
n	n	n	1
1	1	1	1

$\neg$	
1	0
n	n
0	1

The *weak tables* basically differ for the behavior of the third value n and are given by:

$\wedge$	0	n	1
0	0	n	0
n	n	n	n
1	0	n	1

$\vee$	0	n	1
0	0	n	1
n	n	n	n
1	1	n	1

$\neg$	
1	0
n	n
0	1

Each set of tables naturally gives rise to two options for building a three-valued logic, according to the choice of 1 (only) as designated value, or 1 together with the third value n. Therefore, four logics populate the Kleene family:[1]

- Strong Kleene logic [12, §64] and the Logic of Paradox, LP [13], obtained out of the strong Kleene tables by choosing 1 and $1, n$, respectively, as designated values;
- Bochvar's logic [6] and Paraconsistent Weak Kleene logic, PWK [11, 14], given by the weak Kleene tables choosing 1 and $1, n$, respectively, as designated values.

In the present paper we focus on a family of paraconsistent logics including both the Logic of Paradox [13] (LP) and Paraconsistent Weak Kleene logic, PWK [11, 14], which has been recently studied under different perspectives [8, 7].
Different types of sequent calculi has been introduced for LP [1], [2], [3], [5]. On the other hand, to the authors' best knowledge, the only attempt to provide a sequent calculus for PWK is [9]. All the existing sequent calculi for these paraconsistent three-valued logics present non-standard features, for instance non standard axioms [4], logical rules introducing more than one connective [4], [3] or logical rules that can be applied only in presence of certain linguistic conditions (this is the case in [9]). In our approach a *standard Gentzen calculus* for a logic L is a calculus (on multisets) having the following properties:

[1]Here we treat the expression "Kleene family" informally and we do not intend to be exhaustive. There are other logics that could also be considered within the family of Kleene logics, defined by using two or more of these matrices (see for instance [10]).

1. Axioms shall be only of the form $\alpha \Rightarrow \alpha$, for any propositional variable α.
2. The premises of logical rules must contain only *subformulas* of the conclusion and each logical rule must introduce exactly one connective at time.
3. Logical rules must have no linguistic restrictions.
4. Sequents shall be interpreted in the object language, that is: $\Gamma \Rightarrow \Delta$ means that the formula $\bigvee_{i=1}^{n} \delta_i$, with $\delta_i \in \Delta$ follows from the formula $\bigwedge_{j=1}^{m} \gamma_j$, with $\gamma_j \in \Gamma$.
5. Only standard structural rules, i.e. contraction, weakening and cut are (possibly) allowed.

Furthermore, by *quasi-standard* we mean a calculus where condition 4 above is replaced by the usual metalinguistic interpretation of the comma in the sequents.
The main result of this work consists of proving the impossibility of providing *standard*, as well as *quasi-standard* sequent calculi for a family of logics including both LP and PWK.
PWK has been extensively studied with the tools of Abstract Algebraic Logic in [7]. We wonder whether the above mentioned negative result might have an algebraic counterpart.

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