Topological spaces of monadic MV-algebras

Antonio Di Nola[1], Revaz Grigolia[2], and Giacomo Lenzi[1]

[1] University of Salerno, Salerno, Italy
{adinola,gilenzi}@unisa.it
[2] Tbilisi State University, Tbilisi, Georgia
revaz.grigolia@tsu.ge

The finitely valued propositional calculi, which have been described by Łukasiewicz and Tarski in [1], are extended to the corresponding predicate calculi. The predicate Łukasiewicz (infinitely valued) logic QL is defined in the following standard way. The existential (universal) quantifier is interpreted as supremum (infimum) in a complete MV-algebra. Then the valid formulas of predicate calculus are defined as all formulas having value 1 for any assignment. The functional description of the predicate calculus is given by Rutledge in [2]. Scarpellini in [3] has proved that the set of valid formulas is not recursively enumerable.

Let L and L_m denote a first-order language and propositional language, respectively, based on $\cdot, +, \rightarrow, \neg, \exists$. We fix a variable x in L, associate with each propositional letter p in L_m a unique monadic predicate $p^*(x)$ in L and define by induction a translation $\Psi : Form(L_m) \rightarrow Form(L)$ by putting: i) $\Psi(p) = p^*(x)$ if p is propositional variable, ii) $\Psi(\alpha \circ \beta) = \Psi(\alpha) \circ \Psi(\beta)$, where $\circ = \cdot, +, \rightarrow$, iii) $\Psi(\exists\alpha) = \exists x \Psi(\alpha)$.

Monadic MV-algebras were introduced and studied by Rutledge in [2] as an algebraic model for the predicate calculus QL of Łukasiewicz infinite-valued logic, in which only a single individual variable occurs. Rutledge followed P.R. Halmos' study of monadic Boolean algebras. In view of the incompleteness of the predicate calculus the result of Rutledge in [2], showing the completeness of the monadic predicate calculus, has been of great interest.

The characterization of monadic MV-algebras as pair of MV-algebras, where one of them is a special kind of subalgebra (m-relatively complete subalgebra), is given in [4]. MV-algebras were introduced by Chang in [5] as an algebraic model for infinitely valued Łukasiewicz logic. An MV-algebra is an algebra $A = (A, \oplus, \odot, ^*, 0, 1)$ where $(A, \oplus, 0)$ is an abelian monoid, and the following identities hold for all $x, y \in A$: $x \oplus 1 = 1$, $x^{**} = x$, $0^* = 1$, $x \oplus x^* = 1$, $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$, $x \odot y = (x^* \oplus y^*)^*$. An algebra $A = (A, \oplus, \odot, ^*, \exists, 0, 1)$ (for short $(A, \exists)$) is said to be a monadic MV-algebra (MMV-algebra for short) if $A = (A, \oplus, \odot, ^*, 0, 1)$ is an MV-algebra and in addition $\exists$ satisfies the following identities: $x \leq \exists x$, $\exists(x \vee y) = \exists x \vee \exists y$, $\exists(\exists x)^* = (\exists x)^*$, $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$, $\exists(x \odot x) = \exists x \odot \exists x$, $\exists(x \oplus x) = \exists x \oplus \exists x$.

A topological space X is said to be an MV-*space* iff there exists an MV-algebra A such that $Spec(A)$ (= the set of prime filters of the MV-algebra A equipped with spectral topology) and X are homeomorphic. Any MV-space is a Priestley space (X, R) such that $R(x)$ $(= \{y \in X : xRy\}$ is a chain for any $x \in X$ and a morphism between MV-spaces is a strongly isotone map (or an MV-*morphism*), i. e. a continuous map $\varphi : X \rightarrow Y$ such that $\varphi(R(x)) = R(\varphi(x))$ for all $x \in X$.

Define on A the binary relation $\widetilde{\equiv}$ by the following stipulation: $x \widetilde{\equiv} y$ iff $supp^*(x) = supp^*(y)$, where $supp^*(x)$ is defined as the set of all prime filters of A containing the element x [6]. Then, $\widetilde{\equiv}$ is a congruence with respect to $\otimes$ and $\vee$. The resulting set $\beta^*(A) (= A/\widetilde{\equiv})$ of equivalence classes is a bounded distributive lattice (which we also call the Belluce lattice of A) $(\beta^*(A), \vee, \wedge, 0, 1)$, where $\beta^*(x) \wedge \beta^*(y) = \beta^*(x \otimes y)$, $\beta^*(x) \vee \beta^*(y) = \beta^*(x \oplus y) = \beta^*(x \vee y)$, $\beta^*(1) = 1$, $\beta^*(0) = 0$, $\beta^*(x)$ is the equivalence class containing the element x.

Q-distributive lattices were introduced by Cignoli in [7]. A Q-distributive lattice is an algebra $(A, \vee, \wedge, \exists, 0, 1)$ such that $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $\exists$ is a quantifier on A, where: $\exists 0 = 0, a \wedge \exists a = a, \exists(a \wedge \exists b) = \exists a \wedge \exists b, \exists(a \vee b) = \exists a \vee \exists b$.

A $Q-$*space* is a triplet (X, R, E) such that (X, R) is a Priestley space and E is an equivalence relation on X which satisfies the following two conditions: 1) $E(U) \in \mathcal{P}(X)$ for each $U \in \mathcal{P}(X)$, and 2) the equivalence classes for E are closed in X (recall that $E(U)$ is the union of the equivalence classes which intersect U and $\mathcal{P}(X)$ is the set of the clopen increasing subsets of X).

Let (X, R, E) and (Y, S, F) be Q-spaces. A $Q-$ *mapping* from (X, R, E) into (Y, S, F) is a continuous and order-preserving function $f : X \to Y$ such that $E(f^{-1}(V)) = f^{-1}(F(V))$ for each $V \in \mathcal{P}(Y)$. Let $\mathcal{QD}$ and $\mathcal{QD}^*$ be the category of Q-lattices and Q-spaces respectively. There exist contravariant functors $Q^* : \mathcal{QD} \to \mathcal{QD}^*$ and $Q : \mathcal{QD}^* \to \mathcal{QD}$ that define a dual equivalence between $\mathcal{QD}$ and $\mathcal{QD}^*$ [7].

We define a covariant functor γ from the category **MMV** of monadic MV-algebras into the category of Q-distributive lattices $\mathcal{QD}$ in the following way. Let $(A, \exists) \in \mathbf{MMV}$ and define a relative congruence relation $\widetilde{\equiv}_E$ with respect to $\odot, \vee$ and $\exists$ on the $(A, \exists)$: for every $x, y \in A$ $x \widetilde{\equiv}_E y$ if and only if $supp(x) = supp(y)$ and $supp(\exists x) = supp(\exists y)$. Let $\gamma : A \to A/\widetilde{\equiv}$ be a natural mapping. The resulting set $\gamma(A)(= A/\widetilde{\equiv}_E)$ of equivalence classes is a Q-distributive lattice. For each $x \in A$ let us denote by $\gamma(x)$ the equivalence class of x. Let $f : A \to B$ be an MMV-homomorphism. Then $\gamma(f)$ is a Q-mapping from $\gamma(A)$ to $\gamma(B)$ defined as follows: $\gamma(f)(\gamma(x)) = \gamma(f(x))$.

Theorem 1. *If* $(A, \exists) \in \mathbf{MMV}$*, then* $\gamma(A, \exists) \in \mathcal{QD}$*, and* γ *is a covariant functor from the category* **MMV** *into the category of* Q*-distributive lattices* $\mathcal{QD}$*.*

(X, R, E) is named MQ-*space* if (X, R) is an MV-space, (X, R, E) is a Q-space and: $R(E(x)) = E(R(x))$, $E(R^{-1}(x)) = R^{-1}(E(x))$, $R^{-1}(x) \cap E(x) = R(x) \cap E(x) = \{x\}$.

Let $\mathcal{MQ}$ be the category the objects of which are MQ-spaces and morphisms strongly isotone Q-mappings. Strongly isotone Q-mappings we name MQ-*mappings*.

Theorem 2. *There exists a contravariant functor* MQ^* *from* **MMV** *into* $\mathcal{MQ}$*:* $MQ^*(A, \exists) = Q^*(\gamma(A, \exists)) = (\mathcal{F}(A), E(\exists))$*, where* $\mathcal{F}(A)$ *is the prime spectrum of* $\gamma(A, \exists)$ *with the patch topology and the inclusion relation and* $E(\exists) = \{(F, G) \in \mathcal{F}(A)^2 | F \cap \exists\gamma(A, \exists) = G \cap \exists\gamma(A, \exists)\}$.

References

[1] J. Lukasiewicz A. Tarski. Untersuchungen über den aussagenkalkul, Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie, 23, 30–50, 1930.

[2] J. D. Rutledge. A preliminary investigation of the infinitely many-valued predicate calculus, 1959. Ph.D. Thesis, Cornell University.

[3] B. Scarpellini. Die nichtaxiomatisierbarkeit des unendlichwertigen prädikaten-kalkulus von Łukasiewicz, J. Symbolic Logic, 27, 159–170, 1962.

[4] A. Di Nola R. Grigolia. On monadic mv-algebras, Annals of Pure and Applied Logic, 128, 125–139, 2004.

[5] C. C. Chang. Algebraic analysis of many-valued logics, Trans. Amer. Math. Soc., 88, 467-490, 1958.

[6] L. P. Belluce. Semisimple algebras of infinite-valued logic and bold fuzzy set theory, Canad. J. Math., 38, 1356–1379, 1986.

[7] R. Cignoli. Quantifiers on distributive lattices, Discrete Mathematics, 96, 183–197, 1991.