A SIMPLE RESTRICTED PRIESTLEY DUALITY FOR BOUNDED DISTRIBUTIVE LATTICES WITH AN ORDER-INVERTING OPERATION

TOMASZ KOWALSKI

Introduction. Bounded distributive lattices with a single unary order-inverting operation form an algebraic semantics (in a technical sense of Blok-Jónsson equivalence, cf. [4]) for a logic of a minimal negation on top of the classical disjunction and conjunction. This logic was investigated in [8], and found particularly useful for analysing various forms of negation occurring in natural languages. It is quite easy to give a natural sequent system for that logic, and prove cut elimination.

Although Priestley-like dualities for distributive-lattice-based algebras are many and varied, they are either very general and quite complex (e.g., [1] or [3]), or not quite as general as needed here (e.g., [6] or [7]). Canonical extensions, which of course cover our case and a topological duality can be extracted from them (not without some work, see e.g., [5]), are a significantly different setting.

Apart from the connection to the logic of minimal negation, I choose to work with a single unary order-inverting operation only for simplicity. Generalising to any number of unary order-inverting or order-preserving operations is completely straightforward, and generalisations to operations of arbitrary arities should not be difficult either. However, generality and naturalness seem to be contravariant here.

Algebras. Let $\text{BDLN}$ (bounded distributive lattices with negation) stand for the class of all algebras $A = \langle A; \wedge, \vee, \neg, 0, 1 \rangle$ such that $\langle A; \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice, and $\neg$ is a unary operation on $A$ satisfying the quasiequation

$$x \leq y \Rightarrow \neg y \leq \neg x \hspace{1cm} (\star)$$

which states that $\neg$ is an order-inverting operation. It is easily shown that $\text{BDLN}$ is a variety, axiomatised by adding any one of

$$\neg x \vee \neg y \leq \neg(x \wedge y)$$
$$\neg(x \vee y) \leq \neg x \wedge \neg y$$

to the identities defining bounded distributive lattices.

Dual spaces. Some notation first. For a Priestley space $P$, we write $\text{Clup}(P)$ for the set of clopen upsets of $P$. For any ordered set $P$, we write $O(P)$ for the set of downsets (order ideals) of $P$. Any order-preserving map $h : P \rightarrow Q$ between ordered sets $P$ and $Q$ can be naturally lifted to the setwise inverse map $h^{-1} : Q \rightarrow P$. It maps upsets to upsets and downsets to downsets. The lifting can be iterated to $(h^{-1})^{-1} : P(Q) \rightarrow P(Q)$. We will write $\overline{h}$ for this double lifting.

As expected, we will now define a category of Priestley spaces with an additional structure. The objects are pairs $(P, \mathcal{N} : P \rightarrow O(\text{Clup}(P)))$, such that:

1. $P$ is a Priestley space.
2. $\text{Clup}(P)$ is the set of clopen upsets of $P$.
3. $O(\text{Clup}(P))$ is the set of downsets of $\text{Clup}(P)$.
4. $\mathcal{N}(x)$ is a downset for each $x \in P$.
5. $\mathcal{N}(x) \cap \text{Clup}(P)$ is a clopen upset for each $x \in P$.
6. $\mathcal{N}(x) \wedge \mathcal{N}(y) \leq \mathcal{N}(x \wedge y)$ for all $x, y \in P$.
7. $\mathcal{N}(x) \vee \mathcal{N}(y) \leq \mathcal{N}(x \vee y)$ for all $x, y \in P$.
8. $\mathcal{N}(x) \leq \mathcal{N}(y)$ if and only if $x \leq y$ for all $x, y \in P$.
9. $\mathcal{N}(0) = \emptyset$ and $\mathcal{N}(1) = P(\text{Clup}(P))$.
10. $\mathcal{N}(x \wedge \neg y) = \mathcal{N}(x) \cap \mathcal{N}(y)$.
11. $\mathcal{N}(x \vee \neg y) = \mathcal{N}(x) \cup \mathcal{N}(y)$.
12. $\mathcal{N}(\neg x) = \overline{\mathcal{N}(x)}$.
13. $\mathcal{N}(\neg 0) = 0$ and $\mathcal{N}(\neg 1) = P(\text{Clup}(P))$.
14. $\mathcal{N}(P(a)) = \mathcal{N}(a)$ for all $a \in P$.
15. $\mathcal{N}(\neg P(a)) = \overline{\mathcal{N}(a)}$.
16. $\mathcal{N}(P(P(x)))$ is a clopen upset for each $x \in P$.
17. $\mathcal{N}(P((P(x)) \wedge \neg y)) = \mathcal{N}(P(x)) \cap \mathcal{N}(y)$.
18. $\mathcal{N}(P((P(x)) \vee \neg y)) = \mathcal{N}(P(x)) \cup \mathcal{N}(y)$.
19. $\mathcal{N}(P(\neg x)) = \overline{\mathcal{N}(P(x))}$.
20. $\mathcal{N}(P(\neg 0)) = 0$ and $\mathcal{N}(P(\neg 1)) = P(\text{Clup}(P))$.
21. $\mathcal{N}(P(P(x)) \wedge \neg y) = \mathcal{N}(P(x)) \cap \mathcal{N}(y)$.
22. $\mathcal{N}(P(P(x)) \vee \neg y) = \mathcal{N}(P(x)) \cup \mathcal{N}(y)$.
23. $\mathcal{N}(P(\neg P(x))) = \overline{\mathcal{N}(P(x))}$.
24. $\mathcal{N}(P(\neg P(0))) = 0$ and $\mathcal{N}(P(\neg P(1))) = P(\text{Clup}(P))$.
(4) \( \mathcal{N} : P \to \mathcal{O}(\text{Clup}(P)) \) is an order-preserving map, such that for every \( X \in \text{Clup}(P) \), the set \( \{ p \in P : X \in \mathcal{N}(p) \} \) is clopen.

Since the domain and range of the map \( \mathcal{N} : P \to \mathcal{O}(\text{Clup}(P)) \) are completely determined by \( P \), from now on we will write \((P, \mathcal{N}_P)\) for the objects. One may find it convenient to think of \( \mathcal{N} \) as associating a system of non-topological neighbourhoods to any point in \( P \). If \( P \) is finite, then \((P, \mathcal{N}_P)\) is just \( P \) together with an order-preserving map from \( P \) to the set of downsets of \( \text{Clup}(P) \).

If \( P \) is a singleton there are precisely three such objects, and their dual algebras generate the three minimal subvarieties of \( \text{BDLN} \).

Let \((P, \mathcal{N}_P)\) and \((Q, \mathcal{N}_Q)\) be objects, and let \( h : P \to Q \) be a continuous map. Since \( h \) is continuous, the map \( h^{-1} : \text{Clup}(Q) \to \text{Clup}(P) \) is well defined. It follows that the double lifting \( h^{-1} \) is also well defined as a map from \( \mathcal{O}(\text{Clup}(P)) \) to \( \mathcal{O}(\text{Clup}(Q)) \). It is easy to verify that, for a \( W \in \mathcal{O}(\text{Clup}(P)) \), we have \( h(W) = \{ U \in \text{Clup}(Q) : h^{-1}(U) \in W \} \).

Now we can define morphisms. A morphism from \((P, \mathcal{N}_P)\) to \((Q, \mathcal{N}_Q)\) is a continuous map \( h \) such that the diagram

\[
\begin{array}{ccc}
P & \xrightarrow{h} & Q \\
\mathcal{O}(\text{Clup}(P)) & \downarrow{\mathcal{N}_P} & \downarrow{\mathcal{N}_Q} \\
\mathcal{O}(\text{Clup}(Q)) & \xrightarrow{\overline{h}} & \\
\end{array}
\]

commutes. The category we have just defined will be called Priestley neighbourhood systems, or \( \text{PNS} \).

**Theorem 1.** The categories \( \text{BDLN} \) (with homomorphisms) and \( \text{PNS} \) are dually equivalent.

Indeed, this duality is an instance of a restricted Prestley duality, in the sense of [2]. Several existing dualities can be obtained as special cases.

**References**