A SIMPLE RESTRICTED PRIESTLEY DUALITY FOR BOUNDED DISTRIBUTIVE LATTICES WITH AN ORDER-INVERTING OPERATION

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Introduction. Bounded distributive lattices with a single unary order-inverting operation form an algebraic semantics (in a technical sense of Blok-Jónsson equivalence, cf. [4]) for a logic of a minimal negation on top of the classical disjunction and conjunction. This logic was investigated in [8], and found particularly useful for analysing various forms of negation occurring in natural languages. It is quite easy to give a natural sequent system for that logic, and prove cut elimination.

Although Priestley-like dualities for distributive-lattice-based algebras are many and varied, they are either very general and quite complex (e.g., [1] or [3]), or not quite as general as needed here (e.g., [6] or [7]). Canonical extensions, which of course cover our case and a topological duality can be extracted from them (not without some work, see e.g., [5]), are a significantly different setting.

Apart from the connection to the logic of minmal negation, I choose to work with a single unary order-inverting operation only for simplicity. Generalising to any number of unary order-inverting or order-preserving operations is completely straightforward, and generalisations to operations of arbitrary arities should not be difficult either. However, generality and naturalness seem to be contravariant here.

Algebras. Let BDLN (*bounded distributive lattices with negation*) stand for the class of all algebras $\mathbf{A} = \langle A; \wedge, \vee, \neg, 0, 1\rangle$ such that $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice, and $\neg$ is a unary operation on A satisfying the quasiequation

$$x \leq y \Rightarrow \neg y \leq \neg x \tag{$\star$}$$

which states that $\neg$ is an order-inverting operation. It is easily shown that BDLN is a variety, axiomatised by adding any one of

$$\neg x \vee \neg y \leq \neg(x \wedge y)$$
$$\neg(x \vee y) \leq \neg x \wedge \neg y$$

to the identities defining bounded distributive lattices.

Dual spaces. Some notation first. For a Priestley space P, we write $\mathrm{Clup}(P)$ for the set of clopen upsets of P. For any ordered set P, we write $\mathcal{O}(P)$ for the set of downsets (order ideals) of P. Any order-preserving map $h\colon P \to Q$ between ordered sets P and Q can be naturally lifted to the setwise inverse map $h^{-1}\colon \mathcal{P}(Q) \to \mathcal{P}(P)$ taking each $X \in \mathcal{P}(Q)$ to $h^{-1}(X) \in \mathcal{P}(P)$. It maps upsets to upsets and downsets to downsets. The lifting can be iterated to $(h^{-1})^{-1}\colon \mathcal{P}(\mathcal{P}(P)) \to \mathcal{P}(\mathcal{P}(Q))$. We will write $\overline{h}$ for this double lifting.

As expected, we will now define a category of Priestley spaces with an additional structure. The objects are pairs $\big(P,\ \mathcal{N}\colon P \to \mathcal{O}(\mathrm{Clup}(P))\big)$, such that:

(1) P is a Priestley space.
(2) $\mathrm{Clup}(P)$ is the set of clopen upsets of P.
(3) $\mathcal{O}(\mathrm{Clup}(P))$ is the set of downsets of $\mathrm{Clup}(P)$.

(4) $\mathcal{N}\colon P \to \mathcal{O}(\mathrm{Clup}(P))$ is an order-preserving map, such that for every $X \in \mathrm{Clup}(P)$, the set $\{p \in P\colon X \in \mathcal{N}(p)\}$ is clopen.

Since the domain and range of the map $\mathcal{N}\colon P \to \mathcal{O}(\mathrm{Clup}(P))$ are completely determined by P, from now on we will write $(P, \mathcal{N}_P)$ for the objects. One may find it convenient to think of $\mathcal{N}$ as associating a system of non-topological neighbourhoods to any point in P. If P is finite, then $(P, \mathcal{N}_P)$ is just P together with an order-preserving map from P to the set of downsets of (the poset of) upsets of P. If P is a singleton there are precisely three such objects, and their dual algebras generate the three minimal subvarieties of BDLN.

Let $(P, \mathcal{N}_P)$ and $(Q, \mathcal{N}_Q)$ be objects, and let $h\colon P \to Q$ be a continuous map. Since h is continuous, the map $h^{-1}\colon \mathrm{Clup}(Q) \to \mathrm{Clup}(P)$ is well defined. It follows that the double lifting $\overline{h}$ is also well defined as a map from $\mathcal{O}(\mathrm{Clup}(P))$ to $\mathcal{O}(\mathrm{Clup}(Q))$. It is easy to verify that, for a $W \in \mathcal{O}(\mathrm{Clup}(P))$, we have $\overline{h}(W) = \{U \in \mathrm{Clup}(Q)\colon h^{-1}(U) \in W\}$.

Now we can define morphisms. A morphism from $(P, \mathcal{N}_P)$ to $(Q, \mathcal{N}_Q)$ is a continuous map h such that the diagram

$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "\mathcal{N}_P"'] & Q \arrow[d, "\mathcal{N}_Q"] \\ \mathcal{O}(\mathrm{Clup}(P)) \arrow[r, "\overline{h}"] & \mathcal{O}(\mathrm{Clup}(Q)) \end{tikzcd}$$

commutes. The category we have just defined will be called *Priestley neighbourhood systems*, or $\mathbb{PNS}$.

Theorem 1. *The categories* BDLN *(with homomorphisms) and* $\mathbb{PNS}$ *are dually equivalent.*

Indeed, this duality is an instance of a restricted Prestley duality, in the sense of [2]. Several existing dualities can be obtained as special cases.

References

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