A SIMPLE RESTRICTED PRIESTLEY DUALITY FOR BOUNDED DISTRIBUTIVE LATTICES WITH AN ORDER-INVERTING OPERATION

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Introduction. Bounded distributive lattices with a single unary order-inverting operation form an algebraic semantics (in a technical sense of Blok-Jónsson equivalence, cf. [4]) for a logic of a minimal negation on top of the classical disjunction and conjunction. This logic was investigated in [8], and found particularly useful for analysing various forms of negation occurring in natural languages. It is quite easy to give a natural sequent system for that logic, and prove cut elimination.

Although Priestley-like dualities for distributive-lattice-based algebras are many and varied, they are either very general and quite complex (e.g., [1] or [3]), or not quite as general as needed here (e.g., [6] or [7]). Canonical extensions, which of course cover our case and a topological duality can be extracted from them (not without some work, see e.g., [5]), are a significantly different setting.

Apart from the connection to the logic of minmal negation, I choose to work with a single unary order-inverting operation only for simplicity. Generalising to any number of unary order-inverting or order-preserving operations is completely straightforward, and generalisations to operations of arbitrary arities should not be difficult either. However, generality and naturalness seem to be contravariant here.

Algebras. Let BDLN (bounded distributive lattices with negation) stand for the class of all algebras $\mathbf{A} = \langle A; \wedge, \vee, \neg, 0, 1 \rangle$ such that $\langle A; \wedge, \vee, 0, 1 \rangle$ is a bounded distributive lattice, and \neg is a unary operation on A satisfying the quasiequation

$$x \le y \Rightarrow \neg y \le \neg x \tag{(\star)}$$

which states that \neg is an order-inverting operation. It is easily shown that BDLN is a variety, axiomatised by adding any one of

$$\neg x \lor \neg y \le \neg (x \land y)$$
$$\neg (x \lor y) \le \neg x \land \neg y$$

to the identities defining bounded distributive lattices.

Dual spaces. Some notation first. For a Priestley space P, we write $\operatorname{Clup}(P)$ for the set of clopen upsets of P. For any ordered set P, we write $\mathcal{O}(P)$ for the set of downsets (order ideals) of P. Any order-preserving map $h: P \to Q$ between ordered sets P and Q can be naturally lifted to the setwise inverse map $h^{-1}: \mathcal{P}(Q) \to \mathcal{P}(P)$ taking each $X \in \mathcal{P}(Q)$ to $h^{-1}(X) \in \mathcal{P}(P)$. It maps upsets to upsets and downsets to downsets. The lifting can be iterated to $(h^{-1})^{-1}: \mathcal{P}(\mathcal{P}(P)) \to \mathcal{P}(\mathcal{P}(Q))$. We will write \overline{h} for this double lifting.

As expected, we will now define a category of Priestley spaces with an additional structure. The objects are pairs $(P, \mathcal{N}: P \to \mathcal{O}(\text{Clup}(P)))$, such that:

- (1) P is a Priestley space.
- (2) $\operatorname{Clup}(P)$ is the set of clopen upsets of P.
- (3) $\mathcal{O}(\operatorname{Clup}(P))$ is the set of downsets of $\operatorname{Clup}(P)$.

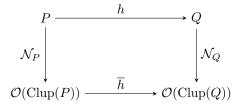
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(4) $\mathcal{N}: P \to \mathcal{O}(\operatorname{Clup}(P))$ is an order-preserving map, such that for every $X \in \operatorname{Clup}(P)$, the set $\{p \in P : X \in \mathcal{N}(p)\}$ is clopen.

Since the domain and range of the map $\mathcal{N}: P \to \mathcal{O}(\operatorname{Clup}(P))$ are completely determined by P, from now on we will write (P, \mathcal{N}_P) for the objects. One may find it convenient to think of \mathcal{N} as associating a system of non-topological neighbourhoods to any point in P. If P is finite, then (P, \mathcal{N}_P) is just P together with an order-preserving map from P to the set of downsets of (the poset of) upsets of P. If P is a singleton there are precisely three such objects, and their dual algebras generate the three minimal subvarieties of BDLN.

Let (P, \mathcal{N}_P) and (Q, \mathcal{N}_Q) be objects, and let $h: P \to Q$ be a continuous map. Since h is continuous, the map $h^{-1}: \operatorname{Clup}(Q) \to \operatorname{Clup}(P)$ is well defined. It follows that the double lifting \overline{h} is also well defined as a map from $\mathcal{O}(\operatorname{Clup}(P))$ to $\mathcal{O}(\operatorname{Clup}(Q))$. It is easy to verify that, for a $W \in \mathcal{O}(\operatorname{Clup}(P))$, we have $\overline{h}(W) = \{U \in \operatorname{Clup}(Q): h^{-1}(U) \in W\}$.

Now we can define morphisms. A morphism from (P, \mathcal{N}_P) to (Q, \mathcal{N}_Q) is a continuous map h such that the diagram



commutes. The category we have just defined will be called *Priestley neighbourhood* systems, or \mathbb{PNS} .

Theorem 1. The categories BDLN (with homomorphisms) and PNS are dually equivalent.

Indeed, this duality is an instance of a restricted Prestley duality, in the sense of [2]. Several existing dualities can be obtained as special cases.

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