Some properties of zero divisor graphs of lattices

Meenakshi P. Wasadikar

Department of Mathematics, Dr. B. A. M. University, Aurangabad, 431004, India
wasadikar@yahoo.com

Beck [3] introduced the notion of coloring in a commutative ring R as follows. Let G be a simple graph whose vertices are the elements of R and two distinct vertices x and y are adjacent in G if $xy = 0$ in R.

Nimbhorkar et al. [8] introduced a graph for a meet-semilattice L with 0, whose vertices are the elements of L and two distinct elements $x, y \in L$ are adjacent if and only if $x \wedge y = 0$. They correlated properties of semilattices with coloring of the associated graph. A nonzero element $a \in L$ is called a zero-divisor if there exists a nonzero $b \in L$ such that $a \wedge b = 0$. We denote by $Z(L)$ the set of all zero-divisors of L. We associate a graph $\Gamma(L)$ to L with vertex set $Z^*(L) = Z(L) - \{0\}$, the set of nonzero zero-divisors of L and distinct $x, y \in Z^*(L)$ are adjacent if and only if $x \wedge y = 0$ and call this graph as the zero-divisor graph of L. In a meet-semilattice L with 0, a nonzero element $a \in L$ is called an atom if there is no $x \in L$ such that $0 < x < a$.

MAIN RESULTS

Lemma 1. *Let L be a complemented distributive lattice. An element $b \in L$ is an atom in L iff $b^{'}$ is the unique end adjacent to b in $\Gamma(L)$.*

Lemma 2. *Let $L \neq C_2$ be a complemented distributive lattice. Then atoms in L are precisely the vertices in $\Gamma(L)$ which are adjacent to an end.*

We recall that C_2 denotes the two element chain.

Lemma 3. *Let $L \neq C_2$ be a complemented distributive lattice. The complement $a^{'}$ of $a \in L$ is also a complement of of a in $\Gamma(L)$. Hence $\Gamma(L)$ is uniquely complemented.*

Lemma 4. *If $\Gamma(L)$ splits into two subgraphs X and Y via a then a is an atom of L.*

However, the converse of Lemma 4 need not hold.

Lemma 5. *If $\Gamma(L)$ splits into two subgraphs X and Y via a then $a \leq x$ for every $x \in L - Z(L)$.*

Lemma 6. *For any lattice L with 0, $L - Z(L)$ is a dual ideal.*

Theorem 1. *Let L be a finite lattice. If $\Gamma(L)$ splits into two subgraphs X and Y via a, then either X or Y is a set of isolated vertices.*

Theorem 2. *If $\Gamma(L)$ splits into two subgraphs X and Y via a, then $N(a)$ is a maximal element in the set $\{N(x) | x \in \Gamma(L)\}$.*

The converse need not hold.

Theorem 3. *If $a - x$ is an edge in $\Gamma(L)$ and a, x are not pendant vertices then the edge $a - x$ is contained in a 3-cycle or a 4-cycle.*

Theorem 4. *Every pair of non-pendant vertices in $\Gamma(L)$ is contained in a cycle of length less than or equal to 6.*

The following example shows that 6 is the best possible bound.

Example 1. *Let L be the lattice of all positive divisors of $n = 4620$ with divisibility as the partial order. Then a, b are adjacent in $\Gamma(L)$ iff the greatest common divisor of a, b is 1. Consider $a = 30$ and $b = 154$. Then a, b are non-pendant vertices in $\Gamma(L)$ and these are contained in the 6-cycle $30 - 7 - 5 - 154 - 3 - 11 - 30$ but not in a cycle of smaller length. Moreover, this cycle is not unique. $30 - 7 - 3 - 154 - 5 - 11 - 30$ is another cycle.*

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