A multi-valued framework for coalgebraic logics over generalised metric spaces

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Introduction. It is by now generally acknowledged that coalgebras for a Set-functor unify a wide variety of dynamic systems [16]. The classical study of their behavior and behavioral equivalence is based on qualitative reasoning – that is, Boolean, meaning that two systems (the systems’ states) are bisimilar (equivalent) or not. But in recent years there has been a growing interest in studying the behavior of systems in terms of quantity. There are situations where one behaviour is smaller than (or, is simulated by) another behaviour, or there is a measurable distance between behaviours in terms of real numbers, as it was done in [15, 18]. This can be achieved by enlarging the coalgebraic set-up to the category of (small) enriched $\mathcal{V}$-categories $\mathcal{V}$-cat [10] ($\mathcal{V}$ is a commutative quantale), which subsumes both ordered sets and (generalised) metric spaces [12].

Coalgebras over generalised metric spaces. The project of developing multi-valued logic for coalgebras on $\mathcal{V}$-cat has started in [1] by extending functors $H : \text{Set} \to \text{Set}$ (and more generally Set-functors which naturally carry a $\mathcal{V}$-metric structure) to $\mathcal{V}$-cat-functors. In this talk, we shall briefly outline the extension procedure: using the density of the discrete functor $D : \text{Set} \to \mathcal{V}$-cat, we apply $H$ to the $\mathcal{V}$-nerve of a $\mathcal{V}$-category, and then take an appropriate quotient in $\mathcal{V}$-cat. If $H$ preserves weak pullbacks, then the above can be obtained using Barr’s relation lifting in a form of “lowest-cost paths” (see also [18, Ch. 4.3], [9]). For example, the extension of the powerset functor yields the familiar Pompeiu-Hausdorff metric, if the quantale is completely distributive.

A logical framework. The next step, following the well-established tradition in coalgebraic logics (see e.g. [14]), is to seek for a contravariant $\mathcal{V}$-cat-enriched adjunction - on top of which to develop coalgebraic logics– involving, on one side, a category of spaces $\text{Sp}$, and on the other side, a category of algebras $\text{Alg}$, obtained eventually by restricting the adjunction $\mathcal{V}$-cat$^{op} \overset{\dashv}{\longrightarrow} \mathcal{V}$-cat. Moreover, we would want for $\text{Alg}$ be a variety in the “world of $\mathcal{V}$-categories”, at least monadic over $\mathcal{V}$-cat. In classical (Boolean) coalgebraic logics (no enrichment), this is achieved by taking $\text{Sp}$ to be Set, and $\text{Alg}$ to be the category of Boolean algebras (see e.g. [7]). One step further, the case of the simplest quantale $\mathcal{V} = 2$ targets positive coalgebraic logics [2], from an order-enriched point of view, by choosing $\text{Sp}$ to be the category of posets and monotone maps, and $\text{Alg}$ to be the category of bounded distributive lattices – which is a finitary ordered variety [4].

In the present work we focus on the unit interval quantale $\mathcal{V} = [0,1]$, endowed with the usual order, the Lukasiewicz tensor given by truncated sum $r \otimes s = \max(0, r + s - 1)$, with

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unit \( e = 1 \) and internal hom (residual) \([r,s] = \min(1-r+s,1)\). Our original motivation to do so came from (at least) the following reason: the unit interval naturally carries an MV-algebra structure. Recall that the MV-algebras are the models for Lukasiewicz multi-valued logic, and that their variety is generated by \([0,1]\) \([5,6]\). As the propositional (Boolean) logic is the base for the usual coalgebraic logic, we looked for a connection between coalgebras based on \([0,1]\)-categories (that is, “bounded-by-1” quasi-metric spaces) and multi-valued logics. However, we shall explain in the talk that MV-algebras are not adequate for our purpose, and propose a different solution instead, detailed below.

An alternative to MV-algebras. The logical connection we therefore propose uses an adaptation of the Priestley duality as in \([8]\). We introduce the notion of a distributive lattice with adjoint pairs of \(\mathcal{V}\)-operators (\(\text{dlao}(\mathcal{V})\)) as a bounded distributive lattice \((A, \land, \lor, 0, 1)\), endowed with a family of adjoint operators \((r \circ - \vdash \land (r, -) : A \to A)_{r \in \mathcal{V}}\), such that the conditions below are satisfied for all \(r, r' \in \mathcal{V}\) and all \(a, a' \in A\):

\[
\begin{align*}
1 \circ a &= a \\
0 \circ a &= 0 \\
\land (1, a) &= a \\
\land (0, a) &= 1 \\
(r \circ r') \circ a &= r \circ (r' \circ a) \\
(r \lor r') \circ a &= (r \circ a) \lor (r' \circ a) \\
\land (r \circ r', a) &= \land (r, \land (r', a)) \\
\land (r \lor r', a) &= \land (r, a) \land \land (r', a)
\end{align*}
\]

Notice that by adjointness \(r \circ - \) preserves finite joins and \(\land (r, -)\) preserves finite meets. A morphism of \(\text{dlao}(\mathcal{V})\) is a bounded distributive lattice map preserving all the adjoint operators \(r \circ -\) and \(\land (r, -)\). Let \(\text{DLatAO}(\mathcal{V})\) be the ordinary category of distributive lattices with adjoint pairs of \(\mathcal{V}\)-operators (notice that \(\text{DLatAO}(\mathcal{V})\) is an algebraic category).

Each \(\text{dlao}(\mathcal{V})\) \(A\) becomes a \(\mathcal{V}\)-category \([3,13]\) with \(\mathcal{V}\)-homs \(A(a,a') = \{r \in [0,1] \mid r \circ a \leq a'\} = \{r \in [0,1] \mid a \leq \land (r, a')\}\), and each \(\text{dlao}(\mathcal{V})\)-morphism is also a \(\mathcal{V}\)-functor. The \(\mathcal{V}\)-categories thus obtained are antisymmetric, finitely complete and cocomplete \([17]\). Consequently, \(\text{DLatAO}(\mathcal{V})\) is a \(\mathcal{V}\text{-cat}\)-category, and it follows that the forgetful functor \(\text{DLatAO}(\mathcal{V}) \to \mathcal{V}\text{-cat}\) is monadic \(\mathcal{V}\text{-cat}\)-enriched.

The ordinary dual category to \(\text{DLatAO}(\mathcal{V})\) can be obtained by adapting the arguments in \([8]\): an object is a Priestley space \((X, \tau, \leq)\), endowed with a family of ternary relations \((R_r)_{r \in \mathcal{V}}\), which satisfy, besides the topological conditions from \([8,\, pp.\, 184-185]\), the requirements that \(R_1\) is the order relation on \(X\), and that \(R_r \circ R_{r'} = R_{r \circ r'}\) and \(R_r \lor R_{r'} = R_{r \lor r'}\) hold. The morphisms are continuous bounded maps \([8, \, Section\, 2.3]\). Denote by \(\text{RelPriest}(\mathcal{V})\) the resulting category. Then the dual equivalence \(\text{RelPriest}(\mathcal{V})^{op} \cong \text{DLatAO}(\mathcal{V})\) is obtained by restricting the usual Priestley duality.

Using the above duality, we can transport the \(\mathcal{V}\text{-cat}\)-category structure on \(\text{RelPriest}(\mathcal{V})\), thus rendering the duality \(\text{RelPriest}(\mathcal{V})^{op} \cong \text{DLatAO}(\mathcal{V})\) \(\mathcal{V}\text{-cat}\)-enriched. The \(\mathcal{V}\text{-cat}\)-category structure such exhibited on \(\text{RelPriest}(\mathcal{V})\) does not say too much at first sight. To gain more insight, we use the lax-algebra framework of \([9]\), in the context of \((\mathcal{V}, \mathcal{V})\)-categories, where \(\mathcal{T}\) is a monad on \(\mathcal{Set}\) which laxly distributes over the \(\mathcal{V}\)-valued powerset monad. We shall see that each relational Priestley space \((X, \tau, \leq, (R_r)_{r \in \mathcal{V}})\) is in fact a \(\mathcal{V}\)-compact topological space \([11]\) — an algebra for the extension of the ultrafilter monad to \(\mathcal{V}\text{-cat}\) (see \([9,\, Ch.\, III.5.2]\) for the cases \(\mathcal{V} = 2\) and \(\mathcal{V} = [0,\infty]\)). The duality \(\text{RelPriest}(\mathcal{V})^{op} \cong \text{DLatAO}(\mathcal{V})\) can now be seen as a \(\mathcal{V}\text{-cat}\)-duality between a category of certain compact \(\mathcal{V}\)-topological spaces (in particular \(\mathcal{V}\)-categories) and a category of algebraic \(\mathcal{V}\)-categories. In future work, more properties of the above duality are planned to be investigated.
References


