A multi-valued framework for coalgebraic logics over generalised metric spaces

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Introduction. It is by now generally acknowledged that coalgebras for a Set-functor unify a wide variety of dynamic systems [16]. The classical study of their behavior and behavioral equivalence is based on qualitative reasoning – that is, Boolean, meaning that two systems (the systems' states) are bisimilar (equivalent) or not. But in recent years there has been a growing interest in studying the behavior of systems in terms of quantity. There are situations where one behaviour is smaller than (or, is simulated by) another behaviour, or there is a measurable distance between behaviours in terms of real numbers, as it was done in [15, 18]. This can be achieved by enlarging the coalgebraic set-up to the category of (small) enriched $\mathscr{V}$-categories $\mathscr{V}\text{-}\mathsf{cat}$ [10] ($\mathscr{V}$ is a commutative quantale), which subsumes both ordered sets and (generalised) metric spaces [12].

Coalgebras over generalised metric spaces. The project of developing multi-valued logic for coalgebras on $\mathscr{V}\text{-}\mathsf{cat}$ has started in [1] by extending functors $H : \mathsf{Set} \to \mathsf{Set}$ (and more generally Set-functors which naturally carry a $\mathscr{V}$-metric structure) to $\mathscr{V}\text{-}\mathsf{cat}$-functors. In this talk, we shall briefly outline the extension procedure: using the density of the discrete functor $D : \mathsf{Set} \to \mathscr{V}\text{-}\mathsf{cat}$, we apply H to the $\mathscr{V}$-nerve of a $\mathscr{V}$-category, and then take an appropriate quotient in $\mathscr{V}\text{-}\mathsf{cat}$. If H preserves weak pullbacks, then the above can be obtained using Barr's relation lifting in a form of "lowest-cost paths" (see also[18, Ch. 4.3], [9]). For example, the extension of the powerset functor yields the familiar Pompeiu-Hausdorff metric, if the quantale is completely distributive.

A logical framework. The next step, following the well-established tradition in coalgebraic logics (see e.g. [14]), is to seek for a contravariant $\mathscr{V}\text{-}\mathsf{cat}$-enriched adjunction - on top of which to develop coalgebraic logics– involving, on one side, a category of spaces Sp, and on the other side, a category of algebras Alg, obtained eventually by restricting the adjunction

$$\begin{tikzcd} \mathscr{V}\text{-}\mathsf{cat}^{\mathsf{op}} \arrow[r, bend left, "{[-,\mathscr{V}]}"] \arrow[r, phantom, "\top"] & \mathscr{V}\text{-}\mathsf{cat} \arrow[l, bend left, "{[-,\mathscr{V}]}"] \end{tikzcd}$$

. Moreover, we would want for Alg be a variety in the "world of $\mathscr{V}$-categories", at least monadic over $\mathscr{V}\text{-}\mathsf{cat}$. In *classical (Boolean) coalgebraic logics* (no enrichment), this is achieved by taking Sp to be Set, and Alg to be the category of Boolean algebras (see e.g. [7]). One step further, the case of the simplest quantale $\mathscr{V} = \mathbb{2}$ targets *positive coalgebraic logics* [2], from an order-enriched point of view, by choosing Sp to be the category of posets and monotone maps, and Alg to be the category of bounded distributive lattices – which is a finitary *ordered* variety [4].

In the present work we focus on the unit interval quantale $\mathscr{V} = [0, 1]$, endowed with the usual order, the Łukasiewicz tensor given by truncated sum $r \otimes s = \max(0, r + s - 1)$, with

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[*] This work has been funded by University Politehnica of Bucharest, through the Excellence Research Grants Program, UPB–GEX, grant ID 252 UPB–EXCELENŢĂ–2016, research project no. 23/26.09.2016.

unit $e = 1$ and internal hom (residual) $[r, s] = \min(1 - r + s, 1)$. Our original motivation to do so came from (at least) the following reason: the unit interval naturally carries an MV-algebra structure. Recall that the MV-algebras are the models for Łukasiewicz multi-valued logic, and that their variety is generated by [0,1] [5, 6]. As the propositional (Boolean) logic is the base for the usual coalgebraic logic, we looked for a connection between coalgebras based on $[0, 1]$-categories (that is, "bounded-by-1" quasi-metric spaces) and multi-valued logics. However, we shall explain in the talk that MV-algebras are not adequate for our purpose, and propose a different solution instead, detailed below.

An alternative to MV-algebras. The logical connection we therefore propose uses an adaptation of the Priestley duality as in [8]. We introduce the notion of a *distributive lattice with adjoint pairs of $\mathscr{V}$-operators* (dlao($\mathscr{V}$)) as a bounded distributive lattice $(A, \wedge, \vee, 0, 1)$, endowed with a family of adjoint operators $(r \odot - \dashv \pitchfork (r, -) : A \to A)_{r\in\mathscr{V}}$, such that the conditions below are satisfied for all $r, r' \in \mathscr{V}$ and all $a, a' \in A$:

$$\begin{aligned} &1 \odot a = a && (r \otimes r') \odot a = r \odot (r' \odot a) \\ &0 \odot a = 0 && (r \vee r') \odot a = (r \odot a) \vee (r' \odot a) \\ &\pitchfork (1, a) = a && \pitchfork (r \otimes r', a) = \pitchfork (r, \pitchfork (r', a)) \\ &\pitchfork (0, a) = 1 && \pitchfork (r \vee r', a) = \pitchfork (r, a) \wedge \pitchfork (r', a) \end{aligned}$$

Notice that by adjointness $r \odot -$ preserves finite joins and $\pitchfork (r, -)$ preserves finite meets. A morphism of dlao($\mathscr{V}$) is a bounded distributive lattice map preserving all the adjoint operators $r\odot-$ and $\pitchfork (r, -)$. Let $\mathsf{DLatAO}(\mathscr{V})$ be the ordinary category of distributive lattices with adjoint pairs of $\mathscr{V}$-operators (notice that $\mathsf{DLatAO}(\mathscr{V})$ is an algebraic category).

Each dlao($\mathscr{V}$) A becomes a $\mathscr{V}$-category [3, 13] with $\mathscr{V}$-homs $A(a, a') = \bigvee\{r \in [0, 1] \mid r \odot a \leq a'\} = \bigvee\{r \in [0, 1] \mid a \leq \pitchfork (r, a')\}$, and each dlao($\mathscr{V}$)-morphism is also a $\mathscr{V}$-functor. The $\mathscr{V}$-categories thus obtained are antisymmetric, finitely complete and cocomplete [17]. Consequently, $\mathsf{DLatAO}(\mathscr{V})$ is a $\mathscr{V}$-cat-category, and it follows that the forgetful functor $\mathsf{DLatAO}(\mathscr{V}) \to \mathscr{V}\text{-}\mathsf{cat}$ is monadic $\mathscr{V}$-cat-enriched.

The ordinary dual category to $\mathsf{DLatAO}(\mathscr{V})$ can be obtained by adapting the arguments in [8]: an object is a Priestley space $(X, \tau, \leq)$, endowed with a family of ternary relations $(R_r)_{r\in\mathscr{V}}$, which satisfy, besides the topological conditions from [8, pp. 184-185], the requirements that R_1 is the order relation on X, and that $R_r \circ R_{r'} = R_{r\otimes r'}$ and $R_r \vee R_{r'} = R_{r\vee r'}$ hold. The morphisms are continuous bounded maps [8, Section 2.3]. Denote by $\mathsf{RelPriest}(\mathscr{V})$ the resulting category. Then the dual equivalence $\mathsf{RelPriest}(\mathscr{V})^{\mathsf{op}} \cong \mathsf{DLatAO}(\mathscr{V})$ is obtained by restricting the usual Priestley duality.

Using the above duality, we can transport the $\mathscr{V}$-cat-category structure on $\mathsf{RelPriest}(\mathscr{V})$, thus rendering the duality $\mathsf{RelPriest}(\mathscr{V})^{\mathsf{op}} \cong \mathsf{DLatAO}(\mathscr{V})$ $\mathscr{V}$-cat-enriched. The $\mathscr{V}$-cat-category structure such exhibited on $\mathsf{RelPriest}(\mathscr{V})$ does not say too much at first sight. To gain more insight, we use the lax-algebra framework of [9], in the context of $(T, \mathscr{V})$-categories, where T is a monad on Set which laxly distributes over the $\mathscr{V}$-valued powerset monad. We shall see that each relational Priestley space $(X, \tau, \leq, (R_r)_{r\in\mathscr{V}})$ is in fact a $\mathscr{V}$-compact topological space [11] – an algebra for the extension of the ultrafilter monad to $\mathscr{V}$-cat (see [9, Ch. III.5.2] for the cases $\mathscr{V} = \mathbb{2}$ and $\mathscr{V} = [0, \infty]$). The duality $\mathsf{RelPriest}(\mathscr{V})^{\mathsf{op}} \cong \mathsf{DLatAO}(\mathscr{V})$ can now be seen as a $\mathscr{V}$-cat-duality between a category of certain compact $\mathscr{V}$-topological spaces (in particular $\mathscr{V}$-categories) and a category of algebraic $\mathscr{V}$-categories. In future work, more properties of the above duality are planned to be investigated.

References

[1] A. Balan, A. Kurz and J. Velebil, Extensions of functors from Set to $\mathscr{V}$-cat. In: L. S. Moss and P. Sobocinski (eds.), *CALCO 2015*, LIPIcs 35:17–34 (2015)

[2] A. Balan, A. Kurz and J. Velebil, Positive fragments of coalgebraic logics. *Logic. Meth. Comput. Sci.* 11(3:18):1–51 (2015)

[3] O. Băbuş, A. Kurz, On the Logic of Generalised Metric Spaces. In: I. Hasuo (ed.), *CMCS 2016*, LNCS 9608:136–155 (2016)

[4] S. L. Bloom, Varieties of ordered algebras. *J. Comput. System Sci.* 13:200–212 (1976)

[5] C. C. Chang, Algebraic analysis of many-valued logics. *Trans. Amer. Math. Soc.* 88:476–490 (1958)

[6] C. C. Chang, A new proof of the completeness of the Łukasiewicz axioms. *Trans. Amer. Math. Soc.* 88:74–80 (1959)

[7] C. Cirstea, A. Kurz, D. Pattinson, L. Schröder and Y. Venema, Modal logics are coalgebraic. Comput. J. 54(1):31–41 (2009)

[8] R. Goldblatt. Varieties of complex algebras. *Ann. Pure Appl. Logic* 44:173–242 (1989)

[9] D. Hofmann, G. J. Seal and W. Tholen, *Monoidal Topology: A Categorical Approach to Order, Metric and Topology.* Encycl. Math. Appl. (2014), Cambridge Univ. Press

[10] G. M. Kelly, *Basic concepts of enriched category theory.* London Math. Soc. LNS 64, Cambridge Univ. Press (1982), Repr. Theory Appl. Categ. 10:1–136 (2005)

[11] H. Lai and W. Tholen, A Note on the Topologicity of Quantale-Valued Topological Spaces, arXiv:cs.LO/1612.09504 (2016)

[12] F. W. Lawvere, Metric spaces, generalised logic, and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano* XLIII:135–166 (1973), Repr. Theory Appl. Categ. 1:1–37 (2002)

[13] D. Hofmann and P. Nora, Enriched Stone-type dualities. arXiv:math.CT/1605.00081 (2016)

[14] C. Kupke and D. Pattinson. Coalgebraic semantics of modal logics: an overview. *Theor. Comput. Sci.* 412(38):5070–5094 (2011)

[15] J. J. M. M. Rutten. Relators and metric bisimulations (extended abstract). In: B. Jacobs et al (eds.), *CMCS 1998, Electr. Notes Theor. Comput. Sci.* 11:1–7 (1998)

[16] J. J. M. M. Rutten, Universal coalgebra: a theory of systems. *Theoret. Comput. Sci.* 249:3–80 (2000)

[17] I. Stubbe, Categorical structures enriched in a quantaloid: tensored and cotensored categories. *Theory Appl. Categ.* 16(14):283–306 (2006)

[18] J. Worrell, On coalgebras and final semantics. PhD thesis, University of Oxford (2000), available at `http://www.cs.ox.ac.uk/people/james.worrell/thesis.ps`